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BASILEAE MCMXXXVIII

VENDITIONI EXPONUNT
ORELL FÜSSLI TURICI ET LIPSIAE
B. G. TEUBNER LIPSIAE ET BEROLINI

LEONHARDI EULERI

COMMENTATIONES ANALYTICAE

AD THEORIAM AEQUATIONUM DIFFERENTIALIUM PERTINENTES

EDIDIT

HENRI DULAC

VOLUMEN POSTERIUS

AUCTORITATE ET IMPENSIS
SOCIETATIS SCIENTIARUM NATURALIUM HELVETICAE

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INVESTIGATIO FUNCTIONUM EX DATA DIFFERENTIALIUM CONDITIONE

Commentatio 285 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 9 (1762/3), 1764, p. 170—212
Summarium ibidem p. 22—23

SUMMARIUM

In superiori volumine[1]) a Cel. Auctore huius dissertationis iam novus quasi campus Analyseos infinitorum est detectus, in quo colendo Geometrae vires suas ad summum universae matheseos sublimioris incrementum exercere queant. Postquam enim observasset omnia praecepta, quae vulgo circa differentiationem et integrationem tradi solent, ad functiones tantum unicae variabilis referri, ita ut etiamsi plures variabiles in calculo occurrant, tamen semper per eliminationem negotium eo perduci debeat, ut tandem ad aequationem duas tantum variabiles complectentem perveniatur, ex qua, qualis altera alterius sit functio, definiri oporteat, hinc istam Analyseos partem, quae adhuc fere sola est exculta, ita definivit Auctor, ut sit methodus functionem unius variabilis ex data eius differentialium cuiusque gradus conditione investigandi, ex quo secunda Analyseos pars circa functiones binarum variabilium versari est censenda, ita ut ex data quapiam relatione inter eius differentialia eius vera indoles investigari debeat, quae investigatio denuo in partes subdividetur, prout in relationem illam differentialia, vel tantum primi, vel etiam secundi, altiorumve ordinum ingrediantur. Iam in hac dissertatione prima istius subdivisionis elementa stabiliuntur atque variae methodi proferuntur functiones binarum variabilium indagandi ex data quacunque differentialium primi gradus relatione. Quodsi nimirum littera V denotet functionem quamcunque binarum variabilium x et y, quas a se invicem prorsus independentes intelligi oportet, ita ut utramque seorsim per omnes valores variare liceat, geminas inde formulas differentiales nasci, est manifestum, hoc modo indicari solitas $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$, quarum illa ex variatione solius x, haec vero solius y oritur, utraque autem est quantitas finita, et more solito ita per differentiationem reperitur, ut, si differentiatio praebeat $dV = Pdx + Qdy$, ubi sine dubio litterae P et Q

1) Confer Commentationem 269 voluminis praecedentis p. 334. H. D.

iterum erunt certae functiones ipsarum x et y, futurum sit $P = \left(\frac{dV}{dx}\right)$ et $Q = \left(\frac{dV}{dy}\right)$. Nunc igitur omnes quaestiones huc pertinentes ita sunt comparatae, ut, data quacunque relatione inter quantitates x, y, V et P, Q, inde litterae P et Q eliminentur, et aequatio ab illis libera tantum inter x et y et V indagetur, quippe qua indoles functionis V, quemadmodum a binis x et y pendet, declarabitur. Cum autem, quando de functionibus unicae variabilis agitur, plurimae quaestiones adhuc calculi vires superent, tum hoc idem multo magis in his quaestionibus circa duas variabiles usu venit, ut numerus earum, quas quidem resolvere licet, admodum sit limitatus, praesertim hoc tempore, quo ista nova Analyseos pars demum tractari est coepta. Interim tamen methodi, quas Auctor hunc in finem excogitavit, mox uberiorem tractationem polliceri videntur.

1. Si V denotet functionem quamcunque binarum variabilium x et y, eaque differentietur, ut prodeat eius differentiale

$$dV = Pdx + Qdy,$$

tum vero hae duae quantitates P et Q denuo differentientur, sicque proveniat

$$dP = pdx + rdy \quad \text{et} \quad dQ = sdx + qdy,$$

notum est, semper fore $r = s$. Quam proprietatem quoque ita exprimere soleo, ut dicam esse

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Huiusmodi scilicet expressione $\left(\frac{dP}{dy}\right)$ indico, functionem P ita differentiari, ut sola quantitas y pro variabili habeatur, et differentiale resultans per dy dividi, quo pacto quantitas finita a differentialibus libera proveniat, necesse est.

2. Quodsi ergo formula $Pdx + Qdy$ ita fuerit comparata, ut secundum hanc scribendi rationem in ea sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, hinc vicissim concludimus, istam formulam $Pdx + Qdy$ revera oriri ex differentiatione cuiuspiam functionis V ipsarum x et y. Cum autem haec proprietas latissime pateat, plus inde concludere non licet, quam formulam $Pdx + Qdy$ esse integrabilem, neque quicquam per hanc solam conditionem in genere definitur, unde ulla peculiaris proprietas eius functionis, ex cuius differentiatione est nata, colligi posset.

3. Quando autem functio V ad certum quoddam genus refertur, tum posito $dV = Pdx + Qdy$, inter quantitates P et Q, praeter illam generalem affectionem, alia quaedam particularis relatio intercedit. Ita novimus, si V sit functio nullius dimensionis binarum variabilium x et y, tum praeter illam generalem proprietatem, qua est $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, insuper hanc particularem locum semper habere, ut sit $Px + Qy = 0$. Deinde simili modo demonstratum extat, si V sit functio homogenea binarum variabilium x et y, cuius dimensionum numerus sit $= n$, eius differentiale $dV = Pdx + Qdy$ semper ita esse comparatum, ut sit

$$nV = Px + Qy.$$

Quo ergo casu hoc imprimis notatu dignum occurrit, quod formulae differentialis $Pdx + Qdy$ integrale statim assignari possit, cum sit

$$V = \frac{1}{n}(Px + Qy).$$

4. Quae cum sint demonstrata, iam pridem in mentem mihi venit, huiusmodi quaestiones inverso modo tractare, atque in methodum inquirere, cuius ope, si posito $dV = Pdx + Qdy$ compertum sit, esse vel $Px + Qy = 0$, vel $Px + Qy = nV$, vicissim inveniri possit, functionem V vel esse nullius dimensionis, vel esse functionem homogeneam, in qua binae variabiles x et y ubique n dimensiones adimpleant. Scilicet nullo respectu ad illas proprietates iam cognitas habito, ex hoc solo, quod sit vel $Px + Qy = 0$, vel $Px + Qy = nV$, per legitima analyseos ratiocinia elici oportet, functionem finitam V hac proprietate praeditam esse, ut vel sit nullius dimensionis, vel homogenea n dimensionum. Intelligendum autem est, proprietatem generalem $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$ semper locum habere debere, sine qua aequatio $dV = Pdx + Qdy$ adeo esset absurda.

5. Hae quaestiones, quae vix adhuc tactae videntur, amplissimum aperiunt campum, fines analyseos ulterius extendendi. Proposita namqu
$dV = Pdx + Qdy$, quaeri potest indoles functionis V
inter binas quantitates P et Q vel adeo inter terna
Huiusmodi autem quaestiones etiamsi paene r
est dubium, quin methodus eas rite resolvendi

allatura sit utilitatem. In problemate enim de cordis vibrantibus tota vis solutionis ad hoc genus est referenda, cum certa quadam relatione inter quantitates P et Q contineatur. Deinde etiam universam motus fluidorum scientiam in huiusmodi formulis differentialibus sum complexus, ubi certa quaedam relatio inter partes differentialium praescribitur, ex qua autem ob defectum talis methodi vix quicquam concludere licet.

6. Huiusmodi autem quaestiones etiam alio modo proponi possunt, ut solius functionis V, cuius natura quaeritur, mentio occurrat. Cum enim, posito $dV = Pdx + Qdy$, sit $P = \left(\frac{dV}{dx}\right)$ et $Q = \left(\frac{dV}{dy}\right)$, prior quaestio ita enunciari poterit:

Invenire indolem functionis V, ut sit

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0\,,$$

posterior vero hoc modo:

Invenire indolem functionis V, ut sit

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = nV\,;$$

ac priori quidem ostendi debet, V esse functionem nullius dimensionis ipsarum x et y; posteriori vero, esse earundem functionem homogeneam n dimensionum. Hoc scribendi autem modo in genere tenendum est, esse

$$dV = dx\left(\frac{dV}{dx}\right) + dy\left(\frac{dV}{dy}\right).$$

latius patens ita se habebit, ut proposita quacunque
$\left(\frac{dV}{x}\right)$ et $\left(\frac{dV}{dy}\right)$ definiri debeat, qualis functio sit V
huiusmodi quaestiones ad functiones trium
ossunt; quin etiam quantitates ex duplici vel
e introduci possunt; cuiusmodi sunt $\left(\frac{ddV}{dx^2}\right)$,
tc., quarum significatus ita se habet, ut verbi

gratia $\left(\frac{d^3V}{dx^2dy}\right)$ oriatur, si primo V differentietur sola x variabili sumta, et differentiale per dx dividatur, ut prodeat $\left(\frac{dV}{dx}\right)$; tum vero haec quantitas denuo differentietur, sola x variabili sumta, ut eius differentiale per dx divisum praebeat $\left(\frac{ddV}{dx^2}\right)$, quod denique rursus differentiatum, sumta sola y variabili, et per dy divisum, dabit $\left(\frac{d^3V}{dx^2dy}\right)$.

8. Dum autem hunc signandi modum recipimus, notandum est, esse

$$\left(\frac{ddV}{dx\,dy}\right)=\left(\frac{ddV}{dy\,dx}\right)$$

et

$$\left(\frac{d^3V}{dx^2dy}\right)=\left(\frac{d^3V}{dx\,dy\,dx}\right)=\left(\frac{d^3V}{dy\,dx^2}\right).$$

Perinde scilicet est, quonam ordine quantitates x et y, quarum alterutra in qualibet differentiatione sola variabilis assumitur, disponantur, dummodo praescriptus differentiationum numerus instituatur. Quin etiam si V sit functio trium variabilium x, y, z, similis convenientia locum habet, erit enim

$$\left(\frac{d^3V}{dx\,dy\,dz}\right)=\left(\frac{d^3V}{dx\,dz\,dy}\right)=\left(\frac{d^3V}{dy\,dx\,dz}\right)=\left(\frac{d^3V}{dy\,dz\,dx}\right)=\left(\frac{d^3V}{dz\,dx\,dy}\right)=\left(\frac{d^3V}{dz\,dy\,dx}\right);$$

hic autem meas investigationes tantum ad functiones duarum variabilium restringam.

9. Hoc modo deducimur non solum ad insolitas et etiamnum vix tractatas quaestiones, sed etiam ad nova signa, quibus adhuc parum sumus adsueti; unde haec methodus, cuius culturam tantopere expetere debemus, non immerito tanquam nova plane analyseos pars est spectanda. Non parum igitur mihi praestitisse videbor, si prima tantum istius methodi fundamenta constituero, neque ob rei novitatem vix minimam aedificii iis superstruendi partem polliceri audeo. Tempore sine dubio haud exiguo et indefesso labore opus erit, antequam istam analyseos partem, in se certe difficillimam, non dicam perficere, sed tantum ad uberiorem usum accommodare liceat. Quam ob rem ea, quae mihi adhuc in hoc genere sunt reperta, ordine ac dilucide exponere constitui,

quo aliis, quos dignitas argumenti ad eundem laborem suscipiendum alliciet, prima quasi obstacula de via removeam, animosque ad novum hoc investigationum genus praeparem.

10. Antequam autem hoc munere fungar, principia quaedam per se perspicua praemittenda sentio. Primo scilicet, si fuerit $\left(\frac{dV}{dx}\right) = 0$, intelligitur, functionem V prorsus non ab x pendere, sed ex sola altera variabili y cum constantibus esse conflatam, sicque etiam formam $\left(\frac{dV}{dy}\right)$ fore functionem ipsius y tantum. Vicissim autem si $\left(\frac{dV}{dy}\right)$ fuerit functio ipsius y tantum, ipsa quantitas V erit aggregatum ex functione ipsius y tantum et ex functione ipsius x tantum; quo ergo casu forma $\left(\frac{dV}{dx}\right)$ erit functio ipsius x tantum. Deinde si fuerit $dV = RdS$, necesse est, ut R sit functio ipsius S, vel S ipsius R, unde et V erit functio ipsius S, vel ipsius R; quia alioquin integrale $\int RdS$ non determinaretur. His igitur positis principiis primum binas quaestiones initio memoratas, quae huic investigationi ansam praebuerunt, resolvam, deinceps ad alias progressurus[1]).

PROBLEMA 1

11. Existente $dV = Pdx + Qdy$, si fuerit $Px + Qy = 0$, invenire, qualis V sit functio ipsarum x et y, ut huic conditioni satisfiat.

SOLUTIO

Cum igitur inter quantitates P et Q haec conditio praescribatur, ut sit $Px + Qy = 0$, erit $Q = -P\frac{x}{y}$, qui valor in aequalitate $dV = Pdx + Qdy$ substitutus dabit

$$dV = P\left(dx - \frac{xdy}{y}\right) = \frac{P(ydx - xdy)}{y};$$

necesse igitur est, ut formula

1) Notandum est magnam partem eorum, quae in hac Commentatione continentur, inveniri in *Institutionum calculi integralis* vol. III § 1—216. LEONHARDI EULERI *Opera omnia*, series I, vol. 13. H. D.

$$\frac{P(ydx - xdy)}{y}$$

sit integrabilis. Quae ut ad formam RdS perducatur, ita repraesentetur

$$dV = Py \cdot \frac{ydx - xdy}{yy}.$$

Sumto enim

$$\frac{ydx - xdy}{yy} = dS$$

et $S = \frac{x}{y}$, cum sit $dV = PydS$, necesse est, ut Py sit functio ipsius S, ideoque et V erit functio ipsius S, hoc est, ipsius $\frac{x}{y}$. Proprietas igitur praescripta, qua est $Px + Qy = 0$, huiusmodi indolem functionis V declarat, ut sit V functio quaecunque ipsius $\frac{x}{y}$; hinc autem manifestum est, pro V prodire functionem nullius dimensionis ipsarum x et y.

COROLLARIUM 1

12. Quodsi ergo ob

$$P = \left(\frac{dV}{dx}\right) \quad \text{et} \quad Q = \left(\frac{dV}{dy}\right)$$

haec proprietas functionis V proponitur, ut sit

$$x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right) = 0,$$

inde certo concludimus, V esse functionem formulae $\frac{x}{y}$ seu, quod eodem redit, esse functionem nullius dimensionis ipsarum x et y.

COROLLARIUM 2

13. Vicissim ergo hinc id, quod quidem iam dudum constat, confirmatur, ut, quoties fuerit V functio nullius dimensionis ipsarum x et y, toties quoque fore

$$x\left(\frac{dV}{dx}\right)+y\left(\frac{dV}{dy}\right)=0.$$

Verum uti hoc facillime ostenditur, ita eius inversum singulari demonstratione indigebat.

SCHOLION

14. Vis huius solutionis in eo est posita, quod differentiale functionis V ad istam formam $dV = RdS$ reduxerim, ex qua, cum unicum differentiale dS contineat, liquido sequebatur, V esse functionem quantitatis S tantum; erat autem $S=\frac{x}{y}$, et notum est, omnes functiones ipsius $\frac{x}{y}$ simul esse functiones nullius dimensionis et vicissim. Eodem ergo principio in solutione sequentium problematum utendum esse intelligetur. Caeterum sine litteris P et Q problema tam proponi, quam resolvi, potuisset: si scilicet quaeri debeat indoles functionis V, ut sit

$$x\left(\frac{dV}{dx}\right)+y\left(\frac{dV}{dy}\right)=0,$$

cum sit

$$dV=dx\left(\frac{dV}{dx}\right)+dy\left(\frac{dV}{dy}\right),$$

ob

$$\left(\frac{dV}{dy}\right)=-\frac{x}{y}\left(\frac{dV}{dx}\right),$$

erit

$$dV=\left(dx-\frac{x\,dy}{y}\right)\cdot\left(\frac{dV}{dx}\right)=y\left(\frac{dV}{dx}\right)\cdot d\cdot\frac{x}{y},$$

manifestum est, V necessario esse debere huius unius quantitatis $\frac{x}{y}$. Quemadmodum autem, si fuerit $dV = Rdr$, recte concluditur, esse V functionem ipsius r tantum, ita porro, si fuerit

$$dV = Rdr + Sds,$$

concludere debemus, V esse functionem binarum variabilium r et s; quod principium utilitatem habebit in indaganda indole functionum trium variabilium, dum quaepiam conditio differentialium proponitur.

PROBLEMA 2

15. Existente

$$dV = Pdx + Qdy,$$

definire indolem functionis V, ut fiat

$$Px + Qy = nV,$$

denotante n numerum quemcunque.

SOLUTIO

Ex conditione praescripta

$$Px + Qy = nV$$

eliciatur altera quantitatum P et Q, puta

$$Q = \frac{nV}{y} - \frac{Px}{y},$$

qui valor in aequalitate differentiali substitutus dabit

$$dV = Pdx + \frac{nVdy}{y} - \frac{Pxdy}{y},$$

cui statim ista forma inducatur:

$$dV - \frac{nVdy}{y} = Py\frac{(ydx - xdy)}{yy} = Pyd\cdot\frac{x}{y},$$

quae, ut prius membrum integrabile reddatur, per y^{-n} multiplicetur, sicque prodibit

$$d\cdot y^{-n}V = Py^{1-n}d\cdot\frac{x}{y},$$

unde concludimus, esse $y^{-n}V$ functionem quantitatis $\frac{x}{y}$, seu functionem nullius dimensionis binarum variabilium x et y. Denotet ergo Z huiusmodi functionem quamcunque nullius dimensionis, et cum sit $y^{-n}V = Z$, erit $V = y^nZ$; talis autem expressio continet omnes functiones homogeneas ipsarum x et y, quarum dimensionum numerus est $= n$.

COROLLARIUM 1

16. Quando ergo noverimus, functionem V eius esse indolis, ut sit

$$nV = x\left(\frac{dV}{dx}\right) + y\left(\frac{dV}{dy}\right),$$

pro certo affirmare poterimus, V esse functionem homogeneam, in qua binae variabiles ubique n dimensiones adimpleant.

COROLLARIUM 2

17. Quodsi ergo ponatur

$$dV = Pdx + Qdy,$$

erunt etiam P et Q functiones homogeneae ipsarum x et y, sed quarum dimensionum numerus est $n-1$, scilicet uno ordine inferior.

COROLLARIUM 3

18. Quare si P et Q fuerint functiones homogeneae binarum variabilium x et y, quarum numerus dimensionum sit idem, puta $= n - 1$, ac si praeterea fuerit

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right),$$

ita ut $Pdx + Qdy$ sit formula differentialis completa, tum eius integrale facillime assignatur. Erit quippe

$$\int(Pdx + Qdy) = \frac{Px + Qy}{n}.$$

Dummodo ergo n non evanescat, integrale huiusmodi formularum sine ulla alia operatione exhiberi potest.

SCHOLION

19. En ergo solutionem ambarum quaestionum, quas initio commemoravi; quae cum iam contineat specimen methodi, qua in hoc genere est utendum,

eandem ad solutionem aliorum similium problematum adhibere licebit. Cum igitur hic proponatur certa quaedam relatio inter functionem V et quantitates inde derivatas $P=\left(\frac{dV}{dx}\right)$ et $Q=\left(\frac{dV}{dy}\right)$, ex qua indolem functionis V definiri oportet, ut ordinem quendam in huiusmodi quaestionibus servem, quoniam tam P et Q, quam V, sunt functiones ipsarum x et y; primum indolem alterius litterarum P et Q dari assumam; deinde ad eiusmodi problemata progrediar, in quibus relatio quaedam inter P et Q praescribitur; tum vero ad talia, ubi vel inter P et V, vel inter Q et V, relatio quaedam intercedere debet. Denique vero relationem datam ad omnes tres quantitates V, P et Q extendi assumam, quemadmodum in problemate secundo est factum. Cum autem hic tantum ad quantitates V, $\left(\frac{dV}{dx}\right)$, $\left(\frac{dV}{dy}\right)$ respiciamus, evidens est, huiusmodi investigationes multo latius extendi posse, dum relatio praescripta alias quantitates ex V derivatas, veluti $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$ et $\left(\frac{ddV}{dy^2}\right)$, complectitur. Verum tantum abest, ut omnium istiusmodi problematum solutiones promittere audeam, ut potius ea tantum, quae sunt faciliora, sim evoluturus. Mox enim patebit, innumerabilia eiusmodi problemata proponi posse, quorum solutiones primos hos conatus longe superent, neque antequam haec quasi nova analyseos pars penitius fuerit exculta, sperari queant.

PROBLEMA 3

20. Existente

$$dV = Pdx + Qdy,$$

si P fuerit functio ipsius x tantum, definire indolem functionis V.

SOLUTIO

Ex parte differentialis Pdx iam functionem V invenire posse notum est, dum spectata y ut constante differentiale Pdx integratur, constans arbitraria autem adiicienda alteram variabilem y utcunque involvere assumitur. Cum igitur P sit functio ipsius x tantum, erit $\int Pdx$ etiam eiusmodi functio, quae sit $= X$, et constans addenda per Y functionem quamcunque ipsius y tantum repraesentetur. Hinc ergo prodibit $V = X + Y$, seu indoles quaesita functionis V in hoc consistet, ut sit V aggregatum duarum functionum, alterius ipsius x tantum, alterius vero ipsius y tantum.

COROLLARIUM

21. Cum ergo hinc fiat

$$dV = dX + dY,$$

manifestum est, si P fuerit functio ipsius x tantum, tum Q fore functionem ipsius y tantum, quae quidem proprietas per se est notissima.

PROBLEMA 4

22. Existente

$$dV = Pdx + Qdy,$$

si P fuerit functio ipsius y tantum, definire indolem functionis V.

SOLUTIO

Cum P sit functio ipsius y tantum, ex sola parte differentialis Pdx, spectata y ut constante, functio V ita definitur, ut sit $V = Px + Y$, denotante Y functionem quamcunque ipsius y. Quare indoles quaesita functionis V in hoc consistet, ut designantibus P et Y functiones quascunque ipsius y, forma functionis V semper sit huiusmodi $V = Px + Y$.

COROLLARIUM 1

23. Si ergo $P = \left(\frac{dV}{dx}\right)$ sit functio ipsius y tantum, cum fiat

$$dV = Pdx + xdP + dY,$$

erit

$$Q = \frac{xdP + dY}{dy},$$

seu ob $\frac{dP}{dy} = \left(\frac{ddV}{dxdy}\right)$, fiet

$$Q = x\left(\frac{ddV}{dxdy}\right) + \frac{dY}{dy}.$$

COROLLARIUM 2

24. Quoties igitur fuerit $\left(\frac{dV}{dx}\right)$ functio ipsius y tantum, toties necesse est, ut sit

$$\left(\frac{dV}{dy}\right) = x\left(\frac{ddV}{dx\,dy}\right) + \frac{dY}{dy},$$

ubi $\frac{dY}{dy}$ denotare potest functionem quamcunque ipsius y tantum. Unde vicissim colligere licet, si fuerit

$$\left(\frac{dV}{dy}\right) = x\left(\frac{ddV}{dx\,dy}\right) + f\cdot y,$$

fore $\left(\frac{dV}{dx}\right)$ functionem ipsius y tantum.

COROLLARIUM 3

25. Simili modo ostendetur, si $Q = \left(\frac{dV}{dy}\right)$ fuerit functio ipsius x tantum, fore

$$V = Qy + X,$$

denotante X functionem quamcunque ipsius x tantum; tum vero etiam, si fuerit

$$\left(\frac{dV}{dx}\right) = y\left(\frac{ddV}{dx\,dy}\right) + f\cdot x,$$

fore $\left(\frac{dV}{dy}\right)$ functionem ipsius x tantum.

PROBLEMA 5

26. Existente $dV = Pdx + Qdy$, si fuerit P functio homogenea ipsarum x et y, cuius dimensionum numerus sit $= n$, definire indolem functionis V.

SOLUTIO

Cum P sit functio homogenea n dimensionum, si pars differentialis Pdx integretur, spectata y ut constante, integrale erit functio homogenea $n + 1$

dimensionum, sit Z talis functio homogenea quaecunque, eritque $V = Z + Y$, denotante Y functionem quamcunque ipsius y tantum, in quo consistit indoles quaesita functionis V.

COROLLARIUM

27. Simili ergo modo ostendetur, si fuerit Q functio homogenea n dimensionum, fore $V = Z + X$, denotante, ut ante, Z functionem homogeneam quamcunque $n+1$ dimensionum, et X ipsius x tantum.

PROBLEMA 6

28. Existente $dV = Pdx + Qdy$, si fuerit $Q = nP$, definire indolem functionis V.

SOLUTIO

Cum sit $Q = nP$, erit $dV = P(dx + ndy)$; quare, posito $x + ny = s$, fiet $dV = Pds$. Ex V valorem certum habere nequit, nisi sit P functio ipsius s, ex qua etiam V erit functio ipsius s. Consequenter si fuerit $Q = nP$, indoles quantitatis V in hoc consistet, ut sit V functio quaecunque formulae $x + ny$; seu[1]) si character Φ adhibeatur ad functionem quamcunque quantitatis, cui praefigitur, designandam, erit $V = \Phi : (x + ny)$.

PROBLEMA 7

29. Existente $dV = Pdx + Qdy$, si fuerit

$$Py + Qx = 0,$$

invenire indolem functionis V.

SOLUTIO

Cum sit $Py + Qx = 0$, erit $Q = -\frac{Py}{x}$, atque hinc

$$dV = Pdx - \frac{Pydy}{x} = \frac{P}{x}(xdx - ydy).$$

1) Vide notam p. 59 voluminis praecedentis. H. D.

Posito ergo $xx - yy = s$, ob $xdx - ydy = \frac{1}{2}ds$, fit $dV = \frac{P}{2x}ds$. Quae formula cum per hypothesin sit integrabilis, necesse est, ut sit $\frac{P}{2x}$ functio ipsius s, unde etiam V prodibit functio ipsius $s = xx - yy$. Quocirca indoles quaesita in hoc consistet, ut V sit functio quaecunque quantitatis $xx - yy$.

PROBLEMA 8

30. Existente $dV = Pdx + Qdy$, si fuerit $Q = Pp$, dum p exprimit functionem quamcunque datam ipsarum x et y, definire indolem functionis V.

SOLUTIO

Habebimus ergo $dV = Pdx + Ppdy = P(dx + pdy)$. Iam consideretur formula $dx + pdy$, quae si non fuerit per se integrabilis, dabitur multiplicator q, qui eam reddat integrabilem. Sit ergo $qdx + pqdy = ds$, eritque s functio quoque data ipsarum x et y, et ob $dx + pdy = \frac{ds}{q}$, habebitur $dV = \frac{P}{q}ds$. Necesse igitur est, ut haec formula sit integrabilis, ideoque indoles quaesita in hoc consistet, ut sit V functio quaecunque quantitatis s, quae quomodo ex x et y sit composita, ex conditione quantitatis datae p colligi debet.

COROLLARIUM 1

31. Problema hoc satis late patet, cum in eo ratio quaecunque inter quantitates P et Q, seu $\left(\frac{dV}{dx}\right)$ et $\left(\frac{dV}{dy}\right)$, proponatur. Si enim sit $P : Q = 1 : p$, quaecunque functio ipsarum x et y pro p fuerit data, qualis futura sit functio V, definiri potest.

COROLLARIUM 2

32. Quanquam autem semper multiplicator q existit, qui formulam $dx + pdy$ integrabilem reddit, tamen saepe evenire potest, ut ob defectum analyseos hic multiplicator assignari nequeat. Atque his casibus solutio problematis impeditur.

COROLLARIUM 3

33. Alio autem loco ostendi, huiusmodi multiplicatores semper exhiberi posse, si aequatio $dx + pdy = 0$ resolvi queat. Quare nisi p eiusmodi fuerit functio ipsarum x et y, ut aequatio $dx + pdy = 0$ resolvi possit, huic analyseos defectui tribui debet, si problema resolvi nequeat.

PROBLEMA 9

34. Si sint X et Y functiones datae, illa ipsius x tantum, haec vero ipsius y tantum, tum vero p sit functio etiam data ipsarum x et y, definire indolem functionis V, ut posito $dV = Pdx + Qdy$ sit $Q = (P + X)\,p + Y$.

SOLUTIO

Cum igitur sit $Q = (P + X)\,p + Y$, erit aequatio differentialis $dV = Pdx + Ppdy + Xpdy + Ydy$, quae ad hanc reducitur formam:

$$dV = (P + X)\,(dx + pdy) - Xdx + Ydy,$$

ubi partes Xdx et Ydy per se sunt integrabiles. Quaeratur ergo iterum multiplicator q formulam $dx + pdy$ integrabilem reddens, sitque $q(dx + pdy) = ds$, atque erit

$$dV = \frac{P+X}{q}\,ds - Xdx + Ydy,$$

quae forma cum per hypothesin sit integrabilis, necesse est, ut $\frac{P+X}{q}$ sit functio ipsius s tantum, quae si ponatur $= S$, prodibit $V = \int Sds - \int Xdx + \int Ydy$, quae est indoles desideratae functionis V.

COROLLARIUM 1

35. Cum igitur ex data functione p definiatur functio s, si pro Σ capiatur functio quaecunque huius quantitatis s, functio quaesita V ita debet esse comparata, ut sit

$$V = \Sigma - \int Xdx + \int Ydy.$$

COROLLARIUM 2

36. Haec igitur adiectio functionum X et Y solutionem problematis non reddidit difficiliorem; dummodo X sit functio ipsius x tantum, et Y ipsius y tantum. Verum solutio, ut ante, pendet a resolutione aequationis differentialis $dx + pdy = 0$, quae si vires nostras superet, etiam problema resolvere non valemus.

COROLLARIUM 3

37. Possunt etiam loco X et Y aliae functiones binarum variabilium x et y, puta M et N, assumi, dummodo formula $Mdx + Ndy$ integrationem admittat. Si enim haec conditio proponatur, ut sit $Q - N = (P - M)\,p$, functio quaesita V ita prodibit expressa:

$$V = \Sigma + \int (Mdx + Ndy).$$

PROBLEMA 10

38. Existente $dV = Pdx + Qdy$, invenire, qualis sit V functio ipsarum x et y, ut fiat

$$Q = \frac{Py}{x} + nx.$$

SOLUTIO

Cum sit

$$Q = \frac{Py}{x} + nx,$$

erit

$$dV = \frac{P}{x}(xdx + ydy) + nxdy.$$

Statuatur $xx + yy = ss$, ut sit $x = \sqrt{(ss - yy)}$, ac fiet

$$dV = \frac{Ps}{x}ds + nxdy = \frac{Ps}{x}ds + ndy\sqrt{(ss - yy)},$$

unde patet, V esse functionem ipsarum y et s, et quidem talem, ut posita s constante, ea differentiata praebeat $ndy\sqrt{(ss - yy)}$. Quare vicissim functio V

reperietur, si formula $ndy\sqrt{(ss-yy)}$, spectato s ut constante, integretur, et insuper functio quaecunque ipsius s adiiciatur. Cum igitur sit

$$\int ndy\sqrt{(ss-yy)} = \tfrac{1}{2}ny\sqrt{(ss-yy)} + \tfrac{1}{2}nss\,\mathrm{A\ sin.}\,\frac{y}{s}\,,$$

erit Φ pro signo functionis cuiuscunque assumendo

$$V = \tfrac{1}{2}nxy + \tfrac{1}{2}n(xx+yy)\,\mathrm{A\ tang.}\,\frac{y}{x} + \Phi : (xx+yy)\,,$$

in qua forma functio quaesita V semper debet contineri.

SCHOLION

39. Hoc exemplum, utut valde particulare, tamen non continetur in problemate praecedente, neque in eius amplificatione ipsi in corollario postremo illata, quoniam in formula reducta membrum

$$nxdy = ndy\sqrt{(ss-yy)}$$

non est integrabile. Quare probe notetur artificium, quo hic sum usus, et quod in hoc consistit, quod valorem differentialis dV ad duas alias variabiles s et y, scilicet $dV = Rds + Tdy$, revocavi, cuius alterum membrum Tdy absolute datur, unde problema ad primum genus pertinet, in quo binarum quantitatum P et Q alterutra est cognita. Huiusque artificii ope problema sequens multo latius patens resolvi poterit.

PROBLEMA 11

40. Si sint p et t functiones datae quaecunque binarum variabilium x et y, definire indolem functionis V, ut posito $dV = Pdx + Qdy$ sit $Q = Pp + t$.

SOLUTIO

Habebitur ergo, substituto pro Q isto valore

$$dV = P(dx + pdy) + tdy\,,$$

ubi pro formula differentiali $dx + pdy$ iterum idoneus multiplicator q quaeratur, qui eam integrabilem reddat. Sit ergo $q\,(dx + pdy) = ds$, et iam quantitas s tanquam nova variabilis introducatur, per quam et y altera variabilis x definiatur. Hoc modo x aequabitur cuipiam functioni datae ipsarum s et y, quae in t ubique loco x scribatur, sicque fiet t functio quoque data ipsarum s et y. Cum ergo ob $dx + pdy = \frac{ds}{q}$ sit $dV = \frac{P}{q}ds + tdy$, erit V eiusmodi functio ipsarum s et y, quae spectata s ut constante differentiata praebeat tdy, quare vicissim pro functione V invenienda integretur formula differentialis tdy, spectata s ut constante; sit integrale hoc modo proveniens $\int tdy = T$, quod igitur etiam datur, tum, quia quantitas P non datur, erit $V = T + \varPhi : s$. Denique hic pro s et in T restituatur valor ipsius s vi x et y, atque patebit, quomodo functio V ex x et y sit composita.

EXEMPLUM

41. *Quaeratur indoles functionis V, ut posita $dV = Pdx + Qdy$, sit $Px + Qy = n\,(xx + yy)$.*

Cum ergo sit

$$Q = -\frac{Px}{y} + \frac{n(xx + yy)}{y}, \quad \text{erit} \quad p = -\frac{x}{y} \quad \text{et} \quad t = \frac{n(xx + yy)}{y},$$

unde

$$dx + pdy = dx - \frac{xdy}{y}.$$

Capiatur $q = \frac{1}{y}$, erit

$$ds = \frac{dx}{y} - \frac{xdy}{yy} \quad \text{et} \quad s = \frac{x}{y}$$

hincque

$$x = sy \quad \text{et} \quad t = ny(ss + 1).$$

Quare spectata s ut constante, habebitur

$$\int tdy = \tfrac{1}{2}nyy(ss + 1) = \tfrac{1}{2}n(xx + yy) = T;$$

sicque tandem prodit

$$V = \tfrac{1}{2}n\,(xx + yy) + \varPhi : \frac{x}{y},$$

ubi notandum est, $\varPhi : \frac{x}{y}$ exprimere functionem quamcunque nullius dimensionis binarum variabilium x et y.

SCHOLION

42. Feliciter igitur expedivimus casum, quo relatio inter P et Q per aequationem quamcunque primi gradus exprimitur, in qua scilicet quantitates P et Q non ultra primam dimensionem assurgunt. Ex tali enim aequatione Q semper ita definietur, ut sit $Q = Pp + t$, existentibus p et t functionibus quibuscunque datis ipsarum x et y. Verum hic iterum aqua haeret, quoties aequationem $dx + pdy = 0$ resolvere non licet, quia tum quantitas s inveniri nequit. Tum vero etiamsi haec quantitas s sit inventa, cum fuerit imprimis transcendens, ex ea plerumque difficillimum erit, variabilem x definire, ita ut tantum binae s et y in calculo supersint. Poterunt quidem subsidia excogitari, quibus tametsi ex functione data t variabilis x non eliminetur, tamen formulae tdy id integrale T erui queat, quod prodire debet, spectata quantitate s ut constante. Verum quantaecunque sint istae difficultates, eae non huic methodo, quam adumbrare coepi, sunt imputandae. Videamus ergo quousque nobis progredi liceat, si relatio inter P et Q per aequationem vel secundi, vel superiorum graduum detur.

PROBLEMA 12

43. Existente $dV = Pdx + Qdy$, definire indolem functionis V, ut fiat $PQ = \alpha$.

SOLUTIO

Cum sit $Q = \frac{\alpha}{P}$, erit $dV = Pdx + \frac{\alpha dy}{P}$, quaeriturque, qualis functio debeat esse P, ut ista formula differentialis $Pdx + \frac{\alpha dy}{P}$ fiat integrabilis. Adhibeamus hic transformationem integralium obviam, qua est $\int zdu = zu - \int udz$, ac reperietur:

$$V = Px + \frac{\alpha y}{P} - \int xdP + \int \frac{\alpha ydP}{PP}.$$

Quare necesse est, ut haec formula differentialis $dP\left(\frac{\alpha y}{PP} - x\right)$ integrabilis existat; id quod fieri nequit, nisi $\frac{\alpha y}{PP} - x$ sit functio ipsius P; quo casu etiam integrale $\int dP\left(\frac{\alpha y}{PP} - x\right)$ fiet functio ipsius P. Denotet ergo Π functionem quamcunque

ipsius P, ac ponatur $\frac{\alpha y}{PP}-x=\Pi$, ex cuius aequationis resolutione quantitas P per x et y definiri intelligitur. Inventa autem hac functione P habebimus

$$V=Px+\frac{\alpha y}{P}+\int \Pi dP\,.$$

COROLLARIUM 1

44. Casus ergo simplicissimus, quo huic problemati satisfit, est si $\Pi=0$, quo fit

$$P=\sqrt{\frac{\alpha y}{x}} \quad \text{et} \quad \int \Pi dP=\text{const.}$$

Habebimus ergo

$$V=2\sqrt{\alpha xy}+\text{Const.},$$

nam ob

$$dV=\frac{dx\sqrt{\alpha y}}{\sqrt{x}}+\frac{dy\sqrt{\alpha x}}{\sqrt{y}}$$

erit utique $PQ=\alpha$.

COROLLARIUM 2

45. Tum sumto $\Pi=\beta$, erit

$$P=\sqrt{\frac{\alpha y}{x+\beta}} \quad \text{et} \quad \int \Pi dP=\beta P\,.$$

Hoc ergo casu consequemur sequentem functionem satisfacientem:

$$V=x\sqrt{\frac{\alpha y}{x+\beta}}+\sqrt{\alpha y(x+\beta)}+\beta\sqrt{\frac{\alpha y}{x+\beta}}=2\sqrt{\alpha y(x+\beta)}\,,$$

et generalius satisfacere manifestum est

$$V=2\sqrt{\alpha\,(x+\beta)\,(y+\gamma)}.$$

COROLLARIUM 3

46. Si velimus functiones magis compositas, quae tamen exhiberi queant, sit

$$\Pi=\beta PP,\ \text{ideoque}\ \int \Pi dP=\tfrac{1}{3}\beta P^3.$$

At cum habeamus

$$\frac{\alpha y}{PP}-x=\beta PP \quad \text{seu} \quad P^4=\frac{-xPP+\alpha y}{\beta},$$

fiet

$$PP=\frac{-x+\sqrt{(xx+4\alpha\beta y)}}{2\beta},$$

unde ob

$$V=\frac{PPx+\alpha y+\frac{1}{3}\beta P^4}{P}=\frac{2PPx+4\alpha y}{3P}$$

et substitutione absoluta

$$V=\sqrt{\frac{2}{9\beta}}\left((xx+4\alpha\beta y)^{\frac{3}{2}}+x(12\alpha\beta y-x^2)\right).$$

SCHOLION 1

47. Eodem modo resolvi posse patet problema, si Q debeat esse functio quaecunque ipsius P. Ponatur enim $dQ=R\,dP$, et fiet

$$V=Px+Qy-\int(x+Ry)\,dP.$$

Oportet ergo esse $x+Ry$ functionem ipsius P, cuius etiam R est functio data. Quare si ponatur, ut ante, $x+Ry=\Pi$, ex hac aequatione P per x et y definietur; quo valore deinceps in Q, R et Π substituto obtinebitur functio $V=Px+Qy-\int\Pi\,dP$ per solas binas variabiles x et y expressa.

EXEMPLUM

48. *Quaeratur functio V, ut posito $dV=Pdx+Qdy$, sit $PP+QQ=aa$, seu $Q=\sqrt{(aa-PP)}$;*

hincque $$R=\frac{-P}{\sqrt{(aa-PP)}} \quad \text{et} \quad x-\frac{Py}{\sqrt{(aa-PP)}}=\Pi,$$

unde P debet definiri. Sumatur autem $\Pi=0$, fiet

$$P=\frac{ax}{\sqrt{(xx+yy)}}, \quad Q=\frac{ay}{\sqrt{(xx+yy)}} \quad \text{et} \quad V=a\sqrt{(xx+yy)}.$$

SCHOLION 2

49. At si Q non solum per P detur, sed etiam variabiles x et y utcunque in eius determinationem ingrediantur, tum hoc modo negotium non expedire licet. Verum his casibus perpendendum est, quemadmodum P et Q ut functiones ipsarum x et y considerantur, ita quaternarum P, Q, x et y binas quasque ut functiones binarum reliquarum considerari posse. Quare quovis casu oblato non ad hanc formulam $Pdx + Qdy$ sumus adstricti, quam integrabilem reddi oporteat, sed negotium pariter conficiemus, si vel hanc $-xdP + Qdy$, vel hanc $Pdx - ydQ$, vel hanc $-xdP - ydQ$ integrabilem reddamus; quin etiam per substitutiones novae variabiles introduci possunt, quo pacto methodus solvendi vehementer amplificabitur; cuius rei aliquot specimina attulisse iuvabit.

PROBLEMA 13

50. Existente $dV = Pdx + Qdy$, si Q detur utcunque per x et P, invenire indolem functionis V.

SOLUTIO

Cum Q ponatur dari per x et P, habebitur aequatio inter ternas quantitates x, P et Q, ex quo etiam P definiri poterit per x et Q, ita ut P aequetur certae cuipiam functioni ipsarum x et Q. Sumantur ergo x et Q pro binis variabilibus, ex quibus reliqua omnia sint determinanda, et cum sit

$$V = Qy + \int (Pdx - ydQ),$$

necesse est, ut formula differentialis $Pdx - ydQ$ sit integrabilis, cuius integrale spectetur tanquam functio ipsarum x et Q. Cum igitur P per x et Q detur, quarta autem variabilis y indefinita relinquatur, hoc integrale $\int (Pdx - ydQ)$ invenietur, si spectata Q ut constante formula Pdx integretur, et ad integrale functio quaecunque ipsius Q adiiciatur. Sit igitur integrale hoc modo sumtum $\int Pdx = R$, eritque R functio data ipsarum x et Q, unde fiat $dR = Pdx + SdQ$, utraque scilicet quantitate x et Q pro variabili sumta. Quo posito habebimus $\int (Pdx - ydQ) = R + \Phi{:}Q$, et $V = Qy + R + \Phi{:}Q$. Designetur iam differentiale ipsius $\Phi{:}\,Q$ per $dQ.\ \Phi'{:}\,Q$, eritque

$$Pdx - ydQ = Pdx + SdQ + dQ.\ \Phi'{:}\,Q,$$

unde fit $y = -S - \Phi' : Q$. Denique ex hac aequatione $y = -S - \Phi' : Q$ una cum relatione, quae inter Q, P et x intercedit, definiantur per binas variabiles x et y alterae binae P et Q, earumque valores restituti ostendent, qualis functio V debeat esse ipsarum x et y; in quo id ipsum, quod quaeritur, consistit.

COROLLARIUM 1

51. Simili modo, si Q detur per y et P, ita, ut x non ingrediatur in hanc relationem, utendum erit hac forma $V = Px + \int(Qdy - xdP)$, quae, cum Q considerari debeat tanquam functio data ipsarum x et P, paribus operationibus ad integrabilitatem perducetur.

COROLLARIUM 2

52. Quodsi autem, vel P, vel Q, per x et y determinantur, quaestio nihil habet difficultatis. Si enim sit P functio data ipsarum x et y, quaeratur integrale $\int Pdx$, spectata y ut constante, positoque $\int Pdx = R$, erit $V = R + \Phi : y$.

COROLLARIUM 3

53. Quoties ergo relatio inter quantitates P, Q, x et y per huiusmodi aequationem datur, in quam tantum ternae harum quantitatum ingrediantur, indoles functionis V per problemata iam tractata definiri potest.

SCHOLION

54. Ex hoc ergo genere supersunt casus, quibus relatio data omnes quatuor litteras P, Q, x et y continet. At pro hoc iam eum casum evolvimus, quo erat $Q = Pp + t$, existentibus p et t functionibus quibuscunque ipsarum x et y, cuius solutio in problemate 11 est exhibita. Quia vero loco binarum variabilium x et y, sequentia paria simili modo tractari possunt:

I. Si sit $Q = xM + N$,

existentibus M et N functionibus quibuscunque ipsarum P et y.

II. Si sit $P = yM + N$,

existentibus M et N functionibus quibuscunque ipsarum Q et x.

III. Si sit $y = xM + N$,

existentibus M et N functionibus quibuscunque ipsarum P et Q. His scilicet quoque casibus solutio per praecepta § 40 tradita inveniri poterit.

EXEMPLUM

55. *Existente* $dV = Pdx + Qdy$, *oporteat definire functionem* V, *ut sit* $xyPQ = \alpha$.

Cum ergo sit $Q = \frac{\alpha}{Pxy}$, erit

$$dV = Pdx + \frac{\alpha dy}{Pxy},$$

qui casus in nullo tractatorum continetur. Verum posito $ly = u$, cum sit $dV = Pdx + \frac{\alpha du}{Px}$, si u loco y spectemus, hancque formam cum $Pdx + Qdy$ comparemus, erit istud $Q = \frac{\alpha}{Px}$, ideoque per x et P tantum datur, ita ut hoc exemplum reductum sit ad praesens problema. Ne igitur hoc Q cum principali confundatur, cum sit $P = \frac{\alpha}{Qx}$, habebimus scribendo y loco u

$$V = Qy + \int\left(\frac{\alpha dx}{Qx} - ydQ\right),$$

sumta ergo Q constante, erit

$$\int\frac{\alpha dx}{Qx} = R = \frac{\alpha}{Q}\, lx,$$

hincque $S = \frac{-\alpha lx}{QQ}$, unde fit

$$u = ly = \frac{\alpha lx}{QQ} - \Phi' : Q$$

et

$$V = Qly + \frac{\alpha lx}{Q} + \Phi : Q$$

existente $\Phi : Q = \int dQ \cdot \Phi' : Q$, ubi pro $\Phi' : Q$ sumi potest functio quaecunque ipsius Q.

Sit ergo pro casu simplicissimo $\Phi' : Q = 0$, eritque

$$Q = \sqrt{\frac{\alpha lx}{ly}} \quad \text{et} \quad V = 2\sqrt{\alpha lx \cdot ly} + \text{Const.}$$

Ac si sumatur $\Phi' : Q = n - \frac{\alpha m}{QQ}$, fiet

$$\Phi : Q = nQ + \frac{\alpha m}{Q} + C \quad \text{et} \quad ly + n = \frac{\alpha(lx+m)}{QQ},$$

hincque

$$Q = \sqrt{\alpha \frac{lx+m}{ly+n}}$$

et

$$V = 2\sqrt{\alpha(lx+m)(ly+n)} + \text{Const.}$$

PROBLEMA 14

56. Existente $dV = Pdx + Qdy$, definire indolem functionis V, ut fiat $PP + QQ = xx + yy$.

SOLUTIO

Hoc problema in nullo casuum hactenus tractatorum continetur; verumtamen idonea transformatione ad casum facillimum reduci potest. Ponatur enim $PP + QQ = xx + yy = tt$, sitque angulis duobus indefinitis Φ et θ introducendis:

$$P = t \sin.\Phi, \quad Q = t \cos.\Phi, \quad x = t \sin.\theta \quad \text{et} \quad y = t \cos.\theta,$$

ob

$$dx = dt \sin.\theta + td\theta \cos.\theta \quad \text{et} \quad dy = dt \cos.\theta - td\theta \sin.\theta$$

erit

$$dV = tdt(\sin.\Phi \sin.\theta + \cos.\Phi \cos.\theta) - ttd\theta(\cos.\Phi \sin.\theta - \sin.\Phi \cos.\theta)$$

seu

$$dV = tdt \cos.(\theta - \Phi) - ttd\theta \sin.(\theta - \Phi).$$

At est

$$\int tdt \cos.(\theta - \Phi) = \tfrac{1}{2} tt \cos.(\theta - \Phi) + \tfrac{1}{2}\int tt(d\theta - d\Phi) \sin.(\theta - \Phi),$$

unde fit

$$V = \tfrac{1}{2} tt \cos.(\theta - \Phi) - \tfrac{1}{2}\int tt(d\theta + d\Phi) \sin.(\theta - \Phi).$$

Cum igitur haec formula integrabilis esse debeat, necesse est, ut sit $tt \sin.(\theta - \Phi) = \text{funct.}(\theta + \Phi)$. Quare cum sit $tt = xx + yy$ et $\text{tang.}\,\theta = \frac{x}{y}$, hinc angulus Φ per x et y determinabitur, cuius valor substitutus dabit functionem V per x et y expressam. Sit, ut functiones algebraicas simpliciores eliciamus,

$$tt \sin.(\theta - \Phi) = \alpha \sin.(\theta + \Phi) + \beta \cos.(\theta + \Phi),$$

eritque

$$V = \tfrac{1}{2} tt \cos.(\theta - \Phi) + \tfrac{1}{2}\alpha \cos.(\theta + \Phi) - \tfrac{1}{2}\beta \sin.(\theta + \Phi)$$

unde, si eliminetur tt, prodit

$$2V \sin.(\theta - \Phi) = \alpha \sin. 2\theta + \beta \cos. 2\theta = \frac{2\alpha xy - \beta(xx - yy)}{xx + yy}.$$

At evolutis illis angulis, fit

$$ttx \cos.\Phi - tty \sin.\Phi = \alpha x \cos.\Phi + \alpha y \sin.\Phi + \beta y \cos.\Phi - \beta x \sin.\Phi,$$

ideoque

$$\operatorname{tang}.\Phi = \frac{ttx - \alpha x - \beta y}{tty + \alpha y - \beta x}$$

et

$$\sec.\Phi = \frac{\sqrt{(t^6 - 2\alpha tt(xx - yy) - 4\beta ttxy + \alpha\alpha tt + \beta\beta tt)}}{tty + \alpha y - \beta x},$$

sit

$$T = t\sqrt{(t^4 - 2\alpha(xx - yy) - 4\beta xy + \alpha\alpha + \beta\beta)},$$

erit

$$\sin.\Phi = \frac{ttx - \alpha x - \beta y}{T} \quad \text{et} \quad \cos.\Phi = \frac{tty + \alpha y - \beta x}{T},$$

hincque

$$\sin.(\theta - \Phi) = \frac{2\alpha xy - \beta(xx - yy)}{Tt},$$

quo valore substituto, orietur $V = \dfrac{T}{2t}$, ideoque

$$V = \tfrac{1}{2}\sqrt{((xx + yy)^2 - 2\alpha(xx - yy) - 4\beta xy + \alpha\alpha + \beta\beta)},$$

quae functio etiam hoc modo repraesentari potest:

$$V = \tfrac{1}{2}\sqrt{((\alpha - xx + yy)^2 + (\beta - 2xy)^2)}.$$

Casus simplicissimus prodit sumendo $\alpha = 0$ et $\beta = 0$, quo fit

$$V = \tfrac{1}{2}(xx + yy), \text{ et } dV = xdx + ydy.$$

SCHOLION

57. Ex hoc problemate intelligitur, quomodo huiusmodi quaestiones, quae, dum omnes quatuor litterae in relationem praescriptam ingrediuntur, solutu difficillimae videntur, idonea tamen substitutione interdum ad casus iam tractatos reduci queant. Neque vero adhuc modum perspicio, quo in genere, utcunque relatio inter quatuor quantitates P, Q, x et y fuerit comparata, solutio obtineri possit; plurima autem alia huiusmodi exempla afferre possem, in quibus reductio ad casus iam tractatos perfici queat; sed quia hoc argumentum minime exhaurire confido, ad sequentia capita progredior, quando relatio praescripta praeter quantitates P, Q, x et y etiam ipsam functionem quaesitam V complectitur; ubi quidem per se est perspicuum, si relatio inter V, x et y tantum proponeretur, ne quaestionem quidem fore, cum functio V immediate per x et y daretur. Quare ab eiusmodi problematibus exordiar, ubi relatio praescripta praeter functionem V alterutram quantitatum P et Q vel etiam ambas implicat; dum variabiles x et y ipsae vel simul ingrediuntur, vel secus. Facile autem intelligitur, haec problemata multo esse difficiliora praecedentibus.

PROBLEMA 15

58. Existente $dV = Pdx + Qdy$, definire indolem functionis V, ut fiat $P = nV$.

SOLUTIO

Cum sit $P = nV$, erit $dV = nVdx + Qdy$, seu $dV - nVdx = Qdy$. Multiplicetur prius membrum per e^{-nx}, ut fiat integrabile, eiusque integrale $e^{-nx}V = \int e^{-nx}Qdy$ aequari debebit functioni cuicunque ipsius y, quae sit $= Y$. Unde functio quaesita erit $V = e^{nx}Y$.

ALITER

Cum V debeat esse eiusmodi functio ipsarum x et y, ut eius differentiale sit $dV = nVdx + Qdy$, perspicuum est, si functio V differentietur posita y constante, proditurum esse $dV = nVdx$. Quare vicissim ex integratione formulae $dV = nVdx$ functio V invenietur, si y ut constans spectetur, tum vero constans per integrationem ingressa quantitatem y utcunque involvere poterit. At aequatio $dV = nVdx$ integrata praebet

$$lV = nx + lY, \text{ seu } V = e^{nx}Y,$$

ut ante.

COROLLARIUM 1

59. Eodem modo resolvi poterit quaestio latius patens, si P debeat esse functio quaecunque ipsius V. Consideretur enim, spectata y ut constante, haec aequatio differentialis $dV = Pdx$, quae, cum duas tantum variabiles contineat V et x, integretur, constanti autem ingressa quantitas y utcunque involvatur.

COROLLARIUM 2

60. Quia binae variabiles x et y sunt inter se permutabiles, eodem modo resolvitur problema, si Q debeat esse functio quaecunque ipsius V.

PROBLEMA 16

61. Existente $dV = Pdx + Qdy$, definire indolem functionis V, ut P fiat functio quaecunque ipsarum V et x.

SOLUTIO

Cum igitur P detur per V et x, si y ut constantem spectemus, habebimus hanc aequationem $dV = Pdx$ inter duas variabiles x et V. Integretur itaque ea, et loco constantis in aequationem integralem introducatur functio quaecunque ipsius y; hoc modo obtinebitur aequatio inter V, x et y, qua indoles functionis V per x et y definietur.

COROLLARIUM 1

62. Quaecunque igitur relatio inter ternas quantitates V, P et x proponatur, sive ex ea V per x et P, sive P per V et x, sive x per V et P definiatur, solutio problematis semper erit in promtu.

COROLLARIUM 2

63. Ob permutabilitatem variabilium x et y, eodem modo problema solvetur, si relatio quaecunque inter Q, V et y proponatur, neque opus est, ut hunc casum seorsim evolvamus.

EXEMPLUM 1

64. *Posito* $dV = Pdx + Qdy$ *oporteat esse*

$$P = \frac{mV}{x} + nx\,.$$

Spectata ergo y ut constante, erit

$$dV = \frac{mVdx}{x} + nxdx \quad \text{seu} \quad dV - \frac{mVdx}{x} = nxdx\,,$$

cuius integralis est

$$\frac{V}{x^m} = \frac{nx^{2-m}}{2-m} + Y$$

existente Y functione quacunque ipsius y. Quare erit

$$V = x^m Y + \frac{n}{2-m}xx\,,$$

si esset $m = 2$, foret $V = xxY + nxxlx$.

EXEMPLUM 2

65. *Posito* $dV = Pdx + Qdy$ *oporteat esse* $aV = P(aa - xx)$.

Cum ergo sit

$$P = \frac{aV}{aa - xx}\,,$$

erit, sumta y constante

$$dV = \frac{aVdx}{aa - xx} \quad \text{seu} \quad \frac{dV}{V} = \frac{adx}{aa - xx}\,,$$

cuius integrale est

$$lV = \tfrac{1}{2}l\frac{a + x}{a - x} + lY\,,$$

unde habebitur

$$V = Y\sqrt{\frac{a + x}{a - x}}\,,$$

denotante Y functionem quamcunque ipsius y.

PROBLEMA 17

66. Existente $dV = Pdx + Qdy$, definire indolem functionis V, si P fiat functio quaecunque data ipsarum x, y et V.

SOLUTIO

Assumo hic, relationem propositam contineri aequatione quacunque inter quatuor quantitates x, y, V et P; ex qua idcirco P per x, y et V definire liceat. Spectetur igitur y ut quantitas constans, et cum sit $dV = Pdx$, haec aequatio iam duas tantum variabiles x et V involvet. Integretur igitur ea, et loco constantis introducatur functio quaecunque ipsius y, hocque modo prodibit aequatio, naturam functionis V ostendens.

COROLLARIUM

67. Simili ergo modo problema solvetur, si proponatur relatio quaecunque inter quatuor quantitates x, y, Q et V, quo casu hoc tantum discrimen observetur, ut primum quantitas x tanquam constans spectetur.

EXEMPLUM

68. *Posito* $dV = Pdx + Qdy$, *oporteat esse* $V = \frac{Px}{y}$.

Cum igitur sit $P = \frac{Vy}{x}$, erit, sumta y constante

$$dV = \frac{Vydx}{x}, \quad \text{ideoque} \quad lV = ylx + lY,$$

unde prodit functio quaesita $V = x^y Y$.

SCHOLION

69. Quodsi ergo in relationem propositam alterutra tantum quantitatum P et Q ingrediatur, problemata sunt solutu facilia. Verum si utraque quantitas P et Q insit, maior difficultas occurrit, quae quandoque tanta est, ut superari nullo modo queat. Quoniam igitur hoc casu solutionem generalem expectare non licet, nonnulla exempla, quae quidem satis late pateant, percurramus.

PROBLEMA 18

70. Existente $dV = Pdx + Qdy$, invenire indolem functionis V, ut fiat $V = mPx + nQy$.

SOLUTIO

Cum hinc sit

$$Q = \frac{V - mPx}{ny},$$

erit

$$dV - \frac{Vdy}{ny} = Pdx - \frac{mPxdy}{ny} = \frac{P}{ny}(nydx - mxdy)\,.$$

Quaeratur multiplicator, qui formulam $nydx - mxdy$ reddat integrabilem, qui cum sit $\frac{1}{xy}$, ideoque

$$dV - \frac{Vdy}{ny} = \frac{Px}{n}\left(\frac{ndx}{x} - \frac{mdy}{y}\right),$$

ponatur

$$nlx - mly = lz \quad \text{seu} \quad z = \frac{x^n}{y^m};$$

unde fit $x = y^{\frac{m}{n}} z^{\frac{1}{n}}$, qui valor loco x substitui concipiatur. Quare cum sit

$$dV = \frac{Vdy}{ny} + \frac{Pxdz}{nz},$$

quantitas V spectari poterit tanquam functio binarum quantitatum y et z, quae igitur talis esse debet, ut sumta z constante fiat $dV = \frac{Vdy}{ny}$. Hinc ergo integrando prodibit:

$$lV = \frac{1}{n}ly + lZ \quad \text{seu} \quad V = y^{\frac{1}{n}} Z$$

sumta pro Z functione quacunque ipsius $z = \frac{x^n}{y^m}$; sicque habebitur

$$V = y^{\frac{1}{n}} \varPhi : \frac{x^n}{y^m}\,.$$

COROLLARIUM 1

71. Cum sit $\dfrac{x^{\frac{1}{m}}}{y^{\frac{1}{n}}}$ functio ipsius $\dfrac{x^n}{y^m}$, erit etiam

$$V = x^{\frac{1}{m}} \varPhi : \frac{x^n}{y^m}.$$

Tum vero etiam ita exhiberi potest

$$V = x^{\frac{1}{m}} \varPhi : \frac{x^{\lambda n}}{y^{\lambda m}} \quad \text{vel} \quad V = y^{\frac{1}{n}} \varPhi : \frac{x^{\lambda n}}{y^{\lambda m}},$$

sumto pro λ numero quocunque.

COROLLARIUM 2

72. Si sit $m = n$, casus habebitur functionum homogenearum supra tractatus. Sumto enim $\lambda = \frac{1}{n}$, denotabit $\varPhi : \frac{x}{y}$ functionem quamcunque nullius dimensionis ipsarum x et y; et V fiet earundem functio homogenea, cuius dimensionum numerus est $= \frac{1}{n}$.

COROLLARIUM 3

73. Si ponamus in genere

$$x^{\frac{1}{m}} = t \quad \text{et} \quad y^{\frac{1}{n}} = u,$$

tum vero capiamus $\lambda = \frac{1}{mn}$, habebimus

$$V = t\, \varPhi : \frac{t}{u},$$

seu V erit functio homogenea unius dimensionis binarum quantitatum t et u.

SCHOLION

74. Si desideretur, ut sit $V = mP + nQ$, solutio aeque parum habebit difficultatis. Nam ob

$$Q = \frac{V}{n} - \frac{mP}{n}$$

erit

$$dV - \frac{Vdy}{n} = P\left(dx - \frac{mdy}{n}\right).$$

Statuatur $x - \frac{my}{n} = z$, ut sit

$$dV = \frac{Vdy}{n} + Pdz\,;$$

iam spectata z ut constante, erit

$$lV = \frac{y}{n} + \Phi : z\,,$$

ideoque

$$V = e^{\frac{y}{n}} \Phi : (nx - my)\,.$$

At si debeat esse $V = mPy + nQx$, ob

$$Q = \frac{V - mPy}{nx}\,,$$

erit

$$dV = P\left(dx - \frac{mydy}{nx}\right) + \frac{Vdy}{nx}\,.$$

Statuatur $nxx - myy = zz$, ut sit

$$x = \sqrt{\frac{zz + myy}{n}}\,,$$

et cum fiat

$$dV = \frac{Vdy}{\sqrt{n(zz + myy)}} + \frac{P}{nx} zdz\,,$$

spectetur quantitas z ut constans, et ob

$$\frac{dV}{V} = \frac{dy}{\sqrt{(nzz + mnyy)}}\,,$$

erit

$$lV = \frac{1}{\sqrt{mn}} l(y\sqrt{mn} + \sqrt{n(zz + myy)}) + lZ\,,$$

ideoque ob $\sqrt{n(zz + myy)} = nx$, prodibit

$$V = (y\sqrt{m} + x\sqrt{n})^{\frac{1}{\sqrt{mn}}} \Phi : (nxx - myy).$$

Quare si esse debeat $V = Py + Qx$, erit

$$V = (x + y)\, \Phi : (xx - yy).$$

Problema autem sequens omnes huiusmodi casus in se complectetur.

PROBLEMA 19

75. Si p sit functio quaecunque data ipsarum x et y, at M functio quaecunque etiam data ipsarum x, y et V, definire indolem functionis V, ut, posito $dV = Pdx + Qdy$, fiat $Q = Pp + M$.

SOLUTIO

Substituto hoc loco Q valore, habemus

$$dV = Mdy + P(dx + pdy).$$

Quaeratur multiplicator q, formulam $dx + pdy$ integrabilem reddens, sitque $\int q\,(dx + pdy) = z$; unde valor ipsius x definiatur per y et z, isque in M, quatenus x inest, loco x substituatur, quo facto erit

$$dV = M\,dy + \frac{Pdz}{q},$$

sicque V considerari poterit ut functio ipsarum y et z. Spectetur nunc z ut quantitas constans, et cum sit $dV = Mdy$, ubi duae tantum variabiles y et V inesse sunt intelligendae, integretur haec aequatio et loco constantis introducatur functio quaecunque ipsius z; in qua si loco z valor in x et y, scilicet $\int q\,(dx + pdy)$, restituatur, illa aequatio $dV = Mdy$ integrata, exhibebit naturam functionis V, quemadmodum ea a binis variabilibus x et y pendere debet.

EXEMPLUM

76. *Posito* $dV = Pdx + Qdy$, *oportet esse* $V = Pyy + Qxx$.

Est ergo
$$Q = -\frac{Pyy}{xx} + \frac{V}{xx},$$

unde fit

$$dV = \frac{Vdy}{xx} + P\left(dx - \frac{yydy}{xx}\right).$$

Sumatur $q = xx$, erit

$$\int(xxdx - yydy) = z = \tfrac{1}{3}x^3 - \tfrac{1}{3}y^3$$

seu $x^3 = y^3 + 3z$, ideoque $xx = (y^3 + 3z)^{\frac{2}{3}}$. Sumto igitur z constante, habetur

$$\frac{dV}{V} = \frac{dy}{(y^3 + 3z)^{\frac{2}{3}}}.$$

Sit itaque S integrale formulae $\dfrac{dy}{(y^3 + 3z)^{\frac{2}{3}}}$, dum z constans assumitur, et obtinebitur

$$V = e^S \Phi : z = e^S \Phi : (x^3 - y^3),$$

scilicet in S loco z ubique eius valor $\frac{1}{3}x^3 - \frac{1}{3}y^3$ substitui debet.

PROBLEMA 20

77. Existente $dV = Pdx + Qdy$, definire indolem functionis V, ut sit $V = nPQ$.

SOLUTIO

Ergo ob $Q = \dfrac{V}{nP}$ erit

$$dV = Pdx + \frac{Vdy}{nP}.$$

Quo iam V ex posteriori membro possit separari, ponatur $P = R\sqrt{V}$, prodibitque

$$\frac{dV}{\sqrt{V}} = Rdx + \frac{dy}{nR},$$

unde convertendo obtinetur

$$2\sqrt{V} = Rx + \frac{y}{nR} - \int dR\left(x - \frac{y}{nRR}\right).$$

Quare necesse est, ut $x - \frac{y}{nRR}$ sit functio ipsius R tantum; ac tali functione assumta definiri poterit R per x et y, unde etiam functio quaesita V per x et y expressa reperietur.

ALITER

Cum sit $V = nPQ$, eliminetur V, ut habeatur haec aequatio

$$nPdQ + nQdP = Pdx + Qdy,$$

ex qua fit

$$dy = -\frac{Pdx}{Q} + \frac{nPdQ}{Q} + ndP,$$

hincque

$$y = nP + \int \frac{P}{Q}(ndQ - dx).$$

Necesse ergo est, ut $\frac{P}{Q}$ sit functio quantitatis $nQ - x$. Ponatur $nQ - x = z$; sitque $\int \frac{P}{Q} dz = \Phi : z$, erit $\frac{P}{Q} = \Phi' : z$ et

$$y = nP + \Phi : z = nQ\Phi' : z + \Phi : z.$$

At est $V = nQQ\Phi' : z$, unde $Q = \sqrt{\frac{V}{n\Phi' : z}}$; sicque habebuntur hae aequationes:

$$\sqrt{\frac{nV}{\Phi' : z}} = x + z \quad \text{et} \quad y = \Phi : z + \sqrt{nV\Phi' : z},$$

ex quibus conficitur $nV = (x + z)(y - \Phi : z)$, ac si eliminetur quantitas z, orietur functio V per x et y expressa.

COROLLARIUM 1

78. Capiatur z constans, seu $ndQ - dx = 0$, fiet

$$Q = \frac{x + a}{n} \quad \text{et} \quad y = nP - b, \quad \text{seu} \quad P = \frac{y + b}{n},$$

unde oritur

$$V = \frac{(x + a)(y + b)}{n},$$

qui est casus simplicissimus.

COROLLARIUM 2

79. Si statuatur $\Phi' : z = a$, erit $\Phi : z = az + b$, unde fit

$$\sqrt{\frac{nV}{a}} = x + z \quad \text{et} \quad nV = (y - az - b)\sqrt{\frac{nV}{a}} \quad \text{seu} \quad \sqrt{naV} = y - b - az,$$

ex quibus coniunctis nanciscimur $2\sqrt{naV} = ax + y - b$, hincque

$$V = \frac{(ax + y - b)^2}{4na},$$

qui est alter casus simplicissimus.

COROLLARIUM 3

80. Sit $\Phi' : z = \frac{1}{(az + b)^2}$, ut sit $\Phi : z = -\frac{1}{a(az + b)} + c$,

et fiet

$$(az + b)\sqrt{nV} = x + z \quad \text{et} \quad y - c + \frac{1}{a(az + b)} = \frac{\sqrt{nV}}{az + b}$$

seu

$$a(az + b)(y - c) = a\sqrt{nV} - 1;$$

at inde est

$$z = \frac{x - b\sqrt{nV}}{a\sqrt{nV} - 1}, \quad \text{ideoque} \quad az + b = \frac{ax - b}{a\sqrt{nV} - 1};$$

hocque valore substituto

$$a(ax - b)(y - c) = (a\sqrt{nV} - 1)^2,$$

quae evolutio praebet

$$V = \frac{1 + a(ax - b)(y - c) \pm 2\sqrt{a(ax - b)(y - c)}}{naa}.$$

PROBLEMA 21

81. Existente $dV = Pdx + Qdy$, si detur V utcunque per P et Q, definire indolem functionis V, seu quemadmodum V per x et y determinetur.

SOLUTIO

Cum igitur sit V functio binarum quantitatum P et Q, ponatur eius differentiale $dV = MdP + NdQ$, eruntque etiam M et N functiones datae ipsarum P et Q. Quare cum sit

$$MdP + NdQ = Pdx + Qdy,$$

erit

$$dy = -\frac{Pdx}{Q} + \frac{MdP + NdQ}{Q},$$

ideoque

$$y = -\frac{Px}{Q} + \int\left(xd\cdot\frac{P}{Q} + \frac{MdP}{Q} + \frac{NdQ}{Q}\right).$$

Ponatur $P = QS$, qui valor in M et N loco P substitui intelligatur, ita ut iam variabiles Q et S considerandae occurrant, fietque

$$y = -Sx + \int(dS(x+M) + \frac{dQ}{Q}(N+MS)).$$

Cum hic M et N sint functiones datae ipsarum Q et S, sumatur S pro constante, ac ponatur integrale

$$\int\frac{dQ}{Q}(N+MS) = R + \Phi : S$$

erit ergo

$$x + M = \left(\frac{dR}{dS}\right) + \Phi' : S,$$

existente $\Phi : S = \int dS\,\Phi' : S$ et

$$y = MS - S\left(\frac{dR}{dS}\right) - S\Phi' : S + R + \Phi : S.$$

Quia nunc R et M dantur per Q et S, et ob $P = QS$, etiam V detur per Q et S. Si haec relatio cum his binis coniungatur

$$x = -M + \left(\frac{dR}{dS}\right) + \Phi' : S \quad \text{et} \quad y = -Sx + R + \Phi : S,$$

poterunt hinc eliminari binae quantitates S et Q, quo facto prodibit aequatio, qua V determinabitur per x et y.

EXEMPLUM 1

82. *Existente $dV = Pdx + Qdy$, oporteat esse $V = mPP + nQQ$.*

Cum ergo sit $dV = 2mPdP + 2nQdQ$, erit $M = 2mP$, et $N = 2nQ$, seu $M = 2mQS$ ob $P = QS$, ita ut sit $V = QQ(mSS + n)$.

Habebimus ergo

$$N + MS = 2Q(mSS + n),$$

ideoque spectata S ut constante

$$R = \int \frac{dQ}{Q}(N + MS) = 2Q(mSS + n),$$

ac proinde $\left(\frac{dR}{dS}\right) = 4mQS$.

Unde has tres aequationes adipiscimur:

I. $V = QQ(mSS + n)$,

II. $x + 2mQS = 4mQS + \varPhi' : S$ seu $x = 2mQS + \varPhi' : S$,

III. $y + Sx = 2Q(mSS + n) + \varPhi : S$ seu $y = 2nQ + \varPhi : S - S\varPhi' : S$.

Quodsi ex II et III eliminetur Q, erit

IV. $nx - mSy = (mSS + n)\varPhi' : S - mS\varPhi : S$,

ex iisdem vero coniunctis fit $Q = \frac{Sx + y - \varPhi : S}{2(mSS + n)}$, quae cum prima dat

V. $2\sqrt{V(mSS + n)} = Sx + y - \varPhi : S$.

Quare superest, ut ex IV et V eliminetur S, sicque prodibit functio V per x et y expressa.

Sit $\varPhi' : S = a$, erit $\varPhi : S = aS + b$, et

IV. $nx - mSy = na - mbS$

V. $2\sqrt{V(mSS + n)} = Sx + y - aS - b$.

Inde est $S = \frac{n(x - a)}{m(y - b)}$, quo valore substituto, erit

$$2\sqrt{mnV} = \sqrt{(n(x - a)^2 + m(y - b)^2)},$$

hincque

$$V = \frac{n(x - a)^2 + m(y - b)^2}{4mn}.$$

EXEMPLUM 2

83. *Existente* $dV = Pdx + Qdy$, *oporteat esse* $V = \frac{P}{Q}$.

Erit ergo

$$M = \frac{1}{Q}, \quad N = -\frac{P}{QQ} = -\frac{S}{Q}$$

ob $P = QS$ et $V = S$ atque $N + MS = 0$, unde fit $R = 0$. Quare prodit

$$x + \frac{1}{Q} = \Phi' : S \quad \text{et} \quad y + Sx = \Phi : S,$$

et quia est $S = V$, ita functio V per x et y determinatur, ut sit $y + Vx = \Phi : V$.

Ponatur $\Phi : V = \dfrac{\alpha + 2\beta V + \gamma VV}{2\delta + 2\varepsilon V}$, ut fiat

$$2\delta y + 2\varepsilon Vy + 2\delta Vx + 2\varepsilon VVx = \alpha + 2\beta V + \gamma VV,$$

hincque

$$VV = \frac{2V(\delta x + \varepsilon y - \beta) + 2\delta y - \alpha}{\gamma - 2\varepsilon x}$$

et

$$V = \frac{\delta x + \varepsilon y - \beta \pm \sqrt{((\delta x - \varepsilon y)^2 + 2(\alpha\varepsilon - \beta\delta)x + 2(\gamma\delta - \beta\varepsilon)y + \beta\beta - \alpha\gamma)}}{\gamma - 2\varepsilon x};$$

si sit γ et $\varepsilon = 0$, erit

$$V = \frac{2\delta y - \alpha}{2\beta - 2\delta x} \quad \text{seu} \quad V = \frac{y - m}{n - x}.$$

SCHOLION

84. Plures aliae huiusmodi quaestiones proponi ac resolvi possent, sed quia earum solutio iisdem principiis, quibus hactenus sum usus, innititur, iis multiplicandis non immoror, cum allatae iam sufficere videantur, ad elementa huius novae methodi condenda. Nonnulla adhuc adiici possent pro casibus, quibus huiusmodi etiam formulae $\left(\frac{ddV}{dx^2}\right)$, $\left(\frac{ddV}{dxdy}\right)$, $\left(\frac{ddV}{dy^2}\right)$ in relationem propositam ingrediuntur, item quando functio quaerenda per tres pluresve variabiles definire debet; verum ne haec tractatio nimis fiat longa, ea in aliam occasionem reservabo.

RECHERCHES SUR L'INTEGRATION DE L'EQUATION $\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right) + \frac{b}{x}\left(\frac{dz}{dx}\right) + \frac{c}{xx}z$

Commentatio 319 indicis ENESTROEMIANI

Mélanges de philosophie et de mathématique de la société royale de Turin 3 (1762/5), 1766, p. 60—91

1. Cette équation est d'un genre tout à fait différent de celui, auquel le calcul intégral a été appliqué jusqu'ici, puisqu'elle renferme trois variables t, x et z, et qu'il s'agit de déterminer la quantité z par les deux autres t et x, c'est-à-dire qu'on demande quelle fonction de t et x doit être prise pour z, afin que la condition prescrite soit remplie. Mais on cherche principalement une solution complète qui nous découvre à la fois toutes les fonctions possibles, qui étant substituées à la place de z satisfassent à l'équation proposée; comme cette équation contient des différentiels du second degré, son intégrale complète doit nécessairement renfermer deux fonctions absolument indéfinies et arbitraires, comme je l'ai prouvé autres fois.

2. Ce n'est pas une pure spéculation, qui a fourni cette équation, mais elle renferme la solution de plusieurs questions physico-mathématiques de la dernière importance. Le cas, où les quantités b et c sont zéro, n'est que trop connu par les recherches sur les vibrations des cordes, dont le mouvement en général est toujours déterminé par cette équation

$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right),$$

pourvu que la corde ait par tout la même grosseur, et que les excursions soient

quasi infiniment petites. Ensuite la question sur la propagation du son m'ayant conduit à cette équation

$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right) + \frac{4a^2}{x}\left(\frac{dz}{dx}\right),$$

il est assès clair, que l'équation proposée étant plus générale est de la dernière importance.

3. Comme cette équation apartient à un genre de calcul tout particulier, dont les premiers élémens ne sont pas encore assès bien développés, je me propose d'établir ici quelques méthodes, qui peuvent être d'un grand usage dans ce nouveau calcul. Car quoique j'aie donné[1]) les intégrales complètes des deux cas mentionnés, c'est plutôt par une heureuse induction, que j'y ai été conduit, que par une méthode certaine, et lorsque Mrs. BERNOULLI et d'ALEMBERT[2]) ont traité le problème des cordes vibrantes, ils se sont contentés des solutions particulières[3]), sans se soucier trop de toute l'étendue, que la question renferme par sa nature. Or avant que d'entreprendre l'intégration de l'équation générale mise dans le titre, il sera bon de traiter plus en détail l'équation

$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right),$$

et d'exposer quelques méthodes, qui nous conduisent à son intégrale complète.

De l'équation $\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right)$.

4. Si l'on vouloit se contenter de solutions particulières, il seroit fort facile d'en trouver une multitude infinie, toutes les valeurs suivantes satisfont également à cette équation:

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1) L. EULER: Mémoires 119, 140, 305. *De vibratione chordarum exercitatio.* Nova acta erud. 1749, p. 512. *Sur la vibration des cordes.* Mém. Berlin, 4, 1750, p. 69. *De la propagation du son.* Mém. Berlin, 15, 1766, p. 189. *LEONHARDI EULERI Opera omnia*, series II, vol. 5, series III, vol. 1. H. D.

2) DANIEL BERNOULLI (1700—1782): *Réflexions et éclaircissements sur les nouvelles vibrations des cordes exposées dans les mémoires de l'Académie de* 1747 *et* 1748. *Sur le mélange de plusieurs espèces de vibrations qui peuvent exister dans un même système de corps.* Mém. Berlin, 9, 1753, p. 147 et p. 173. *Mémoire sur les vibrations des cordes d'une épaisseur inégale.* Mém. Berlin, 21, 1765. JEAN LE ROND D'ALEMBERT (1717—1783): *Recherches sur la courbe que forme une corde tendue mise en vibration.* Mém. Berlin, 3, 1747, p. 214 et 6, 1750, p. 355. H. D.

3) Voir sur ces questions B. RIEMANN: *Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe.* Habilitationsschrift, Göttingen 1854; Abhand. Gesell. Götting. 13, 1866/7. H. BURKHARDT: Die Ausbildung der Reihenentwicklungen an physikalischen Problemen. Jahresbericht der deutschen Math. Ver., 1901 (1908). H. D.

$$z = A(x \pm at)^n, \quad z = A e^{n(x \pm at)}, \quad z = A \sin. n(x \pm at),$$

et l'on peut encore joindre ensemble autant qu'on voudra de telles valeurs, de sorte que par ce moyen on pourroit donner une solution qui renfermât une infinité de quantités constantes et absolument arbitraires. Cependant une telle solution ne seroit pas encore complète, et ne comprendroit point toutes les solutions possibles. Mais quand j'ai trouvé

$$z = \Gamma : (x + at) + \varDelta : (x - at),$$

où $\Gamma : (x + at)$ marque une fonction quelconque de $x + at$, et $\varDelta : (x - at)$ de $x - at$, on voit bien que cette forme est infiniment plus générale[1]), aussi est-elle l'intégrale complète de notre équation.

5. Il est aisé de s'assurer, que cette formule convient à l'équation proposée; car en marquant le différentiel de $\Gamma : u$ par $du\, \Gamma' : u$, et celui de $\Gamma' : u$ par $du\, \Gamma'' : u$, on en tire

$$\left(\frac{dz}{dx}\right) = \Gamma' : (x + at) + \varDelta' : (x - at) \quad \text{et}$$
$$\left(\frac{dz}{dt}\right) = a\Gamma' : (x + at) - a\varDelta' : (x - at)$$

et par la différentiation réitérée

$$\left(\frac{ddz}{dx^2}\right) = \Gamma'' : (x + at) + \varDelta'' : (x - at) \quad \text{et}$$
$$\left(\frac{ddz}{dt^2}\right) = a^2\Gamma'' : (x + at) + a^2\varDelta'' : (x - at),$$

d'où l'on a évidemment $\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right)$.

Mais il s'agit ici d'une méthode, qui partant de l'équation différentio-différentielle nous conduise immédiatement à la formule $z = \Gamma : (x + at) + \varDelta : (x - at)$, et par laquelle nous puissions être assurés, que cette formule est en effet l'intégrale complète de l'équation différentio-différentielle. Je m'en vais proposer deux méthodes qui conduisent à ce but[2]).

1) Voir la note précédente. H. D.

2) Comparer ci-après § 10 du mémoire 737. Voir § 296—300, *Institutiones calculi integralis*, vol. III. *Leonhardi Euleri Opera omnia*, series I, vol. 23 et 13. H. D.

6. *Première méthode.* Je transforme l'équation par cette substitution

$$\left(\frac{dz}{dt}\right)+a\left(\frac{dz}{dx}\right)=u\,,$$

d'où je tire:

$$\left(\frac{du}{dt}\right)=\left(\frac{ddz}{dt^2}\right)+a\left(\frac{ddz}{dt\,dx}\right) \quad \text{et} \quad \left(\frac{du}{dx}\right)=\left(\frac{ddz}{dt\,dx}\right)+a\left(\frac{ddz}{dx^2}\right),$$

et partant

$$\left(\frac{du}{dt}\right)-a\left(\frac{du}{dx}\right)=\left(\frac{ddz}{dt^2}\right)-aa\left(\frac{ddz}{dx^2}\right)=0\,.$$

Nous voilà donc parvenus à une équation différentielle du premier dégré

$$\left(\frac{du}{dt}\right)=a\left(\frac{du}{dx}\right),$$

dont il s'agit de trouver la fonction u. Or, puisque

$$du=dt\left(\frac{du}{dt}\right)+dx\left(\frac{du}{dx}\right),$$

nous aurons

$$du=\left(\frac{du}{dx}\right)\cdot(a\,dt+dx)\,,$$

laquelle formule devant être intégrable, il faut que $\left(\frac{du}{dx}\right)$ soit une fonction de la quantité $x+at$. Posons donc

$$\left(\frac{du}{dx}\right)=\Gamma'':(x+at)\,,$$

et l'équation

$$du=(dx+a\,dt)\cdot\Gamma'':(x+at)$$

donne

$$u=\Gamma':(x+at).$$

Maintenant nous avons à résoudre cette équation

$$\left(\frac{dz}{dt}\right)+a\left(\frac{dz}{dx}\right)=\Gamma':(x+at)$$

dont je cherche l'intégrale de la manière suivante.

7. Posant

$$dz = p\,dx + q\,dt,$$

pour avoir

$$p = \left(\frac{dz}{dx}\right) \quad \text{et} \quad q = \left(\frac{dz}{dt}\right),$$

à cause de

$$q = -ap + \Gamma' : (x + at),$$

nous aurons

$$dz = p(dx - a\,dt) + dt\,\Gamma' : (x + at).$$

Posons

$$p = r + \alpha\Gamma' : (x + at),$$

pour avoir

$$dz = r(dx - a\,dt) + [\alpha dx + (1 - \alpha a)dt]\,\Gamma' : (x + at),$$

et soit $\frac{1-\alpha a}{\alpha} = a$ ou $\alpha = \frac{1}{2a}$, donc

$$dz = r(dx - a\,dt) + \frac{1}{2a}(dx + a\,dt)\Gamma' : (x + at),$$

et puisque le dernier membre est intégrable, l'intégrale étant $\frac{1}{2a}\Gamma : (x + at)$ ou bien simplement $\Gamma : (x + at)$, pour rendre le premier membre également intégrable, il faut qu'il soit

$$r = \Delta' : (x - at),$$

d'où nous tirons

$$z = \Gamma : (x + at) + \Delta : (x - at)$$

et il est évident en même tems, que cette forme est l'intégrale complète de l'équation proposée.

8. *Autre méthode.* Changeons les variables t et x, dont z doit être une fonction, et posons

$$x + at = p \quad \text{et} \quad x - at = q,$$

de sorte qu'il faille à présent trouver, quelle fonction de p et q doit être z. Il s'agit donc d'éliminer t et x, et de trouver une équation différentielle entre p, q et z, ce qui se fera de la manière suivante. Puisque

$$dz = dt\left(\frac{dz}{dt}\right) + dx\left(\frac{dz}{dx}\right) \quad \text{et} \quad dz = dp\left(\frac{dz}{dp}\right) + dq\left(\frac{dz}{dq}\right),$$

à cause de

$$dp = dx + adt \quad \text{et} \quad dq = dx - adt,$$

nous aurons

$$dt\left(\frac{dz}{dt}\right) + dx\left(\frac{dz}{dx}\right) = (dx + adt)\left(\frac{dz}{dp}\right) + (dx - adt)\left(\frac{dz}{dq}\right).$$

Maintenant parceque t et x ne dépendent point l'une de l'autre, les membres affectés de dx et dt doivent être égales séparément, ce qui donne

$$\left(\frac{dz}{dx}\right) = \left(\frac{dz}{dp}\right) + \left(\frac{dz}{dq}\right) \quad \text{et} \quad \left(\frac{dz}{dt}\right) = a\left(\frac{dz}{dp}\right) - a\left(\frac{dz}{dq}\right)$$

et de là nous tirons

$$\left(\frac{ddz}{dp\,dx}\right) = \left(\frac{ddz}{dp^2}\right) + \left(\frac{ddz}{dpdq}\right) \quad \text{et} \quad \left(\frac{ddz}{dpdt}\right) = a\left(\frac{ddz}{dp^2}\right) - a\left(\frac{ddz}{dpdq}\right)$$
$$\left(\frac{ddz}{dq\,dx}\right) = \left(\frac{ddz}{dpdq}\right) + \left(\frac{ddz}{dq^2}\right) \quad \text{et} \quad \left(\frac{ddz}{dq\,dt}\right) = a\left(\frac{ddz}{dpdq}\right) - a\left(\frac{ddz}{dq^2}\right).$$

9. Pour achever la substitution, aiant

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddz}{dpdx}\right) + \left(\frac{ddz}{dq\,dx}\right) \quad \text{et} \quad \left(\frac{ddz}{dt^2}\right) = a\left(\frac{ddz}{dpdt}\right) - a\left(\frac{ddz}{dq\,dt}\right),$$

on n'a qu'à substituer ici les valeurs, que nous venons de trouver, et nous obtiendrons

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{ddz}{dp^2}\right) + 2\left(\frac{ddz}{dp\,dq}\right) + \left(\frac{ddz}{dq^2}\right) \quad \text{et}$$
$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dp^2}\right) - 2aa\left(\frac{ddz}{dpdq}\right) + aa\left(\frac{ddz}{dq^2}\right).$$

Donc notre équation

$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right)$$

se change en celle-ci:

$$2aa\left(\frac{ddz}{dpdq}\right) = -2aa\left(\frac{ddz}{dpdq}\right)$$

ou bien

$$\left(\frac{ddz}{dpdq}\right) = 0,$$

qui est beaucoup plus simple que la proposée, ne contenant qu'un seul membre, qu'il faut égaler à zéro, d'où l'on comprend de quelle importance peuvent souvent être de telles transformations, dans ce nouveau genre du calcul intégral.

10. Tout se réduit donc à trouver, quelle fonction des deux variables p et q doit être la quantité z, afin que l'on ait $\left(\frac{ddz}{dpdq}\right)=0$. Or cette équation est fort aisée à résoudre; qu'on pose, pour mettre l'opération dans tout son jour,

$$\left(\frac{dz}{dq}\right)=u\,,$$

et puisque $\left(\frac{ddz}{dpdq}\right)=\left(\frac{du}{dp}\right)$, il faut qu'il soit $\left(\frac{du}{dp}\right)=0$, d'où il est clair que la quantité u ne sauroit renfermer la variable p, et partant elle sera fonction de la seule variable q, et une fonction quelconque de q mise pour u satisfera à la condition $\left(\frac{du}{dp}\right)=0$. Posons donc

$$u=\Delta':q\,,$$

ce qui est l'intégrale complète de l'équation $\left(\frac{du}{dp}\right)=0$, et maintenant nous aurons

$$\left(\frac{dz}{dq}\right)=\Delta':q\,.$$

Considérons ici l'autre quantité p constante, pour avoir

$$dz=dq\,\Delta':q\,,$$

et partant son intégrale

$$z=\Delta:q+\text{const.};$$

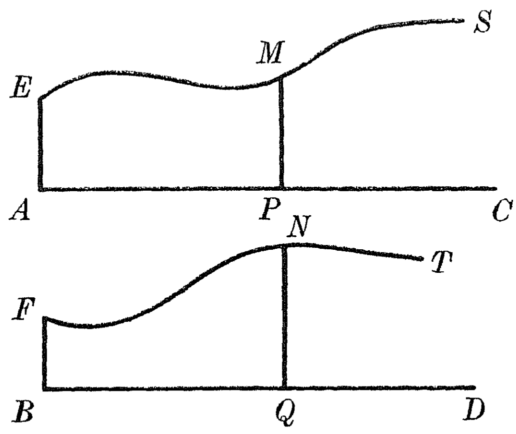
mais cette constante renferme une fonction quelconque de p, au lieu de laquelle écrivant $\Gamma:p$ nous trouvons comme ci-dessus

$$z=\Gamma:p+\Delta:q$$

ou bien

$$z=\Gamma:(x+at)+\Delta:(x-at)\,.$$

11. Pour mettre devant les yeux toute l'étendue de cette solution, on peut décrire à volonté deux courbes quelconques EMS et FNT (fig.) rapportées à leurs axes AC et BD, et alors pour avoir la valeur de z, qui répond à



des valeurs quelconques des deux variables x et t, qu'on prenne dans la première courbe l'abscisse $AP = x + at$, et dans l'autre l'abscisse $BQ = x - at$, et la somme des deux appliquées $PM + QN$ ou bien aussi leur différence représentera en général la nature de la fonction cherchée z qui convient à l'équation

$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right).$$

Cette construction est si générale, que des traits tirés librement sur le papier sans aucune loi peuvent être employés pour les deux lignes EMS et FNT[1]).

12. C'est aussi en quoi consiste le caractère essentiel de ce genre de calcul intégral, et qui le distingue des intégrations ordinaires, où aucune ligne courbe destituée de la loi de continuité, ne sauroit jamais avoir lieu. Mais ici les lignes EMS et FNT peuvent être tirées d'une manière quelconque, sans qu'elles soient comprises dans quelque équation que ce soit; on les peut terminer brusquement où l'on veut, et les composer d'arcs de courbes tout à fait différentes. De quelque

1) Comparer ci-après, p. 76 et 78, les § 3—6 du mémoire 322 de l'Index d'ENESTROEM. H. D.

manière qu'on ait commencé à tirer ces lignes, on est toujours le maître de les continuer de part et d'autre comme on veut, sans qu'on soit astreint à aucune loi; toutes ces irrégularités n'empêchent pas que la construction donnée ne satisfasse aussi bien à l'équation proposée, que si ces deux lignes courbes étoient tout à fait régulières.

De l'équation $\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\alpha}{x}\left(\frac{dz}{dx}\right) + \frac{\beta}{xx}z$.[1]

13. D'abord je remarque qu'on peut transformer cette équation en d'autres semblables, en supposant

$$z = x^\lambda u,$$

d'où l'on a

$$\left(\frac{dz}{dt}\right) = x^\lambda\left(\frac{du}{dt}\right), \quad \left(\frac{dz}{dx}\right) = \lambda x^{\lambda-1}u + x^\lambda\left(\frac{du}{dx}\right),$$

et de là

$$\left(\frac{ddz}{dt^2}\right) = x^\lambda\left(\frac{ddu}{dt^2}\right), \quad \left(\frac{ddz}{dx^2}\right) = \lambda(\lambda-1)x^{\lambda-2}u + 2\lambda x^{\lambda-1}\left(\frac{du}{dx}\right) + x^\lambda\left(\frac{ddu}{dx^2}\right),$$

et partant cette substitution, en divisant par x^λ, nous fournit l'équation

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) + \frac{\alpha+2\lambda}{x}\left(\frac{du}{dx}\right) + \frac{\beta+\alpha\lambda+\lambda(\lambda-1)}{xx}u.$$

Donc si nous prenons

$$\lambda = -\tfrac{1}{2}\alpha, \quad \text{de sorte que} \quad z = x^{-\frac{1}{2}\alpha}u,$$

l'équation proposée se transforme dans celle-ci

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) + \frac{4\beta-\alpha\alpha+2\alpha}{4xx}\cdot u,$$

qui aiant un terme de moins, doit être regardée comme plus simple.

14. Par la même substitution on peut faire évanouir le dernier terme, en posant

1) Comparer: *Institutiones calculi integralis*, vol. III, § 333, 343, 353—356. Voir la note p. 44. H. D.

$$\beta + \alpha\lambda + \lambda\lambda - \lambda = 0,$$

ce qui donne

$$\lambda = \frac{1-\alpha}{2} \pm \sqrt{\left(\frac{(1-\alpha)^2}{4} - \beta\right)} \quad \text{et} \quad \alpha + 2\lambda = 1 \pm \sqrt{[(1-\alpha)^2 - 4\beta]},$$

donc la substitution $z = x^\lambda u$ nous conduit à cette équation

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) + \frac{1 \pm \sqrt{((1-\alpha)^2 - 4\beta)}}{x}\left(\frac{du}{dx}\right).$$

Donc sans restraindre la forme proposée nous pouvons d'abord supposer $\beta = 0$, de sorte que nous aions à résoudre cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\alpha}{x}\left(\frac{dz}{dx}\right),$$

où il est remarquable que posant

$$z = x^{1-\alpha}u,$$

elle se transforme dans celle-ci, qui lui est semblable

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) + \frac{2-\alpha}{x}\left(\frac{du}{dx}\right),$$

et partant si la résolution de l'équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\alpha}{x}\left(\frac{dz}{dx}\right)$$

réussit dans le cas de $\alpha = n$, elle réussira aussi dans le cas de $\alpha = 2 - n$.

15. Mais puisque la destruction du dernier terme pourroit conduire à des valeurs imaginaires, qu'il faudroit donner à l'exposant λ, il vaudra mieux chasser l'avant dernier terme, ce qui se peut toujours faire sans tomber dans cet inconvénient, et partant nos recherches rouleront sur la résolution de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\beta}{xx}z,$$

sur laquelle je remarque, qu'on la peut réduire à une forme où le dernier terme manque d'une double manière, en posant

$$z = x^{\frac{1 \pm \sqrt{(1-4\beta)}}{2}} u\,,$$

d'où l'on parvient à cette forme:

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) + \frac{1 \pm \sqrt{(1-4\beta)}}{x}\left(\frac{du}{dx}\right).$$

Or si β est un nombre positif plus grand que $\frac{1}{4}$, cette forme devient imaginaire.

16. Or par rapport à cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\beta}{xx}z$$

je dois avertir d'avance, que je ne la saurois résoudre en général, quelque nombre qu'on prenne pour β, il n'y a que certaines valeurs de cette lettre, où l'intégration réussisse, en quoi la résolution est semblable à la fameuse équation de Riccati, qui n'est résoluble qu'en certains cas. Mais aussi dans ces cas je dois avouer, que je ne suis pas assès content de la méthode, qui m'a conduit à l'intégration de ces cas, puisqu'elle m'a été fournie par la considération d'une solution particulière, comme on peut le voir dans mes Recherches sur la propagation du son. Cependant j'expliquerai cette méthode, toute imparfaite qu'elle puisse paroître encore, peut être qu'elle donnera à d'autres occasion de mieux approfondir les mystères, que ce nouveau genre de calcul intégral renferme.

17. Après plusieurs essais j'ai trouvé que l'intégrale de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\beta}{xx}z$$

peut être représentée par une telle suite indéfinie:

$$z = Ax^n\Gamma:(x \pm at) + Bx^{n+1}\Gamma':(x \pm at) + Cx^{n+2}\Gamma'':(x \pm at) + Dx^{n+3}\Gamma''':(x \pm at) + \text{etc.},$$

d'où nous tirons d'abord

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = Ax^n\Gamma'':(x \pm at) + Bx^{n+1}\Gamma''':(x \pm at) + Cx^{n+2}\Gamma^{IV}:(x \pm at) + Dx^{n+3}\Gamma^{V}:(x \pm at) + \text{etc.}$$

pour la première partie de l'équation; or pour l'autre nous trouvons

$$\begin{aligned}\left(\frac{ddz}{dx^2}\right) &= n(n-1)Ax^{n-2}\Gamma:(x\pm at) + 2nAx^{n-1}\Gamma':(x\pm at) + Ax^n\Gamma'':(x\pm at)\\ &\qquad\qquad\qquad\qquad + (n+1)nB \qquad\qquad + 2(n+1)B\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + (n+2)(n+1)C\\ \frac{\beta}{xx}z &= \beta A \qquad\qquad\qquad\qquad + \beta B \qquad\qquad\qquad + \beta C\end{aligned}$$

qui devant être égalée à celle-là, nous en tirons les déterminations suivantes

$$\begin{array}{lll} n(n-1)+\beta = 0 & \text{ou} & \beta = -n(n-1)\\ 2nA + 2nB = 0 & \text{ou} & B = -A\\ 2(n+1)B + 2(2n+1)C = 0 & \text{ou} & (n+1)B + 2(n+\frac{1}{2})C = 0\\ 2(n+2)C + 2(3n+3)D = 0 & \text{ou} & (n+2)C + 3(n+\frac{2}{2})D = 0\\ 2(n+3)D + 2(4n+6)E = 0 & \text{ou} & (n+3)D + 4(n+\frac{3}{2})E = 0\\ 2(n+4)E + 2(5n+10)F = 0 & \text{ou} & (n+4)E + 5(n+\frac{4}{2})F = 0.\end{array}$$

18. Voici donc les déterminations de tous nos coéficiens, en supposant $\beta = -n(n-1)$.

$$B = -A$$

$$C = -\frac{(n+1)B}{2(n+\frac{1}{2})} = +\frac{(n+1)}{2(n+\frac{1}{2})}A$$

$$D = -\frac{(n+2)C}{3(n+1)} = -\frac{(n+2)}{2\cdot 3(n+\frac{1}{2})}A$$

$$E = -\frac{(n+3)D}{4(n+\frac{3}{2})} = +\frac{(n+2)(n+3)}{2\cdot 3\cdot 4(n+\frac{1}{2})(n+\frac{3}{2})}A$$

$$F = -\frac{(n+4)E}{5(n+2)} = -\frac{(n+3)(n+4)}{2\cdot 3\cdot 4\cdot 5(n+\frac{1}{2})(n+\frac{3}{2})}A$$

$$G = -\frac{(n+5)F}{6(n+\frac{5}{2})} = +\frac{(n+3)(n+4)(n+5)}{2\cdot 3\cdot 4\cdot 5\cdot 6(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})}A$$

$$H = -\frac{(n+6)G}{7(n+3)} = -\frac{(n+4)(n+5)(n+6)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7(n+\frac{1}{2})(n+\frac{3}{2})(n+\frac{5}{2})}A$$

etc. etc.

19. De là nous voions que toutes les fois que n est un nombre entier négatif, la série supposée devient finie, et partant c'est dans ces cas, que nous

pouvons assigner l'intégrale de notre équation. Posons donc $n = -i$, marquant par i un nombre entier quelconque, et l'équation, dont nous sommes en état de trouver l'intégrale, sera

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{i(i+1)}{xx} z$$

de laquelle il est bon de remarquer, que posant

$$z = x^{i+1} u,$$

elle se change dans cette forme:

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) + \frac{2(i+1)}{x}\left(\frac{du}{dx}\right).$$

Or posant

$$z = x^{-i} u$$

elle se transforme dans celle-ci:

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) - \frac{2i}{x}\left(\frac{du}{dx}\right)$$

d'où nous voions que cette forme:

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{\alpha}{x}\left(\frac{dz}{dx}\right)$$

est intégrable toutes les fois, que α est un nombre pair, soit positif, soit négatif.

1r CAS $n = -1$

Ou intégration de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2}{xx} z.$$

20. Puisque $n = -1$, nous aurons $B = -A$, et les coéficiens suivans s'évanouiront tous, donc l'intégrale de cette équation sera, en prenant $A = 1$, et partant $B = -1$,

$$z = \frac{1}{x}\Gamma : (x \pm at) - \Gamma' : (x \pm at)$$

et tenant compte du signe ambigu de la quantité a, l'intégrale complète sera exprimée ainsi:

$$z=\frac{1}{x}\Gamma:(x-at)-\Gamma':(x-at)+\frac{1}{x}\Delta:(x+at)-\Delta':(x+at)$$

et de la même manière on pourra aisément trouver les intégrales complètes pour tous les autres cas intégrables. Aiant trouvé les coéficiens de la forme supposée, on n'a qu'à résoudre chaque terme en deux, en employant pour l'un la forme $x+at$, et pour l'autre la forme $x-at$.

2^d CAS $n=-2$

Ou intégration de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{6}{xx}z.$$

21. Aiant toujours $B=-A$ à cause de $n=-2$, nous aurons $C=\frac{1}{3}A$; prenant $A=3$, et partant $B=-3$, et $C=1$, l'intégrale complète de notre équation sera

$$z=\frac{3}{x^2}\Gamma:(x+at)-\frac{3}{x}\Gamma':(x+at)+\Gamma'':(x+at)$$
$$+\frac{3}{x^2}\Delta:(x-at)-\frac{3}{x}\Delta':(x-at)+\Delta'':(x-at),$$

où Γ et Δ signifient des fonctions quelconques, exprimées par des appliquées de lignes courbes quelconques, tant régulières qu'irrégulières, et puisque ces deux fonctions sont absolument indépendantes l'une de l'autre, c'est en quoi consiste le caractère de la solution complète.

3^e CAS $n=-3$

Ou intégration de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{12}{xx}z.$$

22. Ici le quatrième coéficient D entre aussi dans l'intégrale avec les trois précédens, dont voici les valeurs

$$B = -A\,,\quad C = -\tfrac{2}{5}B = \tfrac{2}{5}A\,,\quad D = -\tfrac{1}{6}C = -\frac{1\cdot 2}{5\cdot 6}A\,.$$

Posons donc pour éviter les fractions

$$A = 3\cdot 5,\quad B = -3\cdot 5,\quad C = 2\cdot 3,\quad D = -1$$

et l'intégrale complète sera

$$z = \frac{3\cdot 5}{x^3}\Gamma:(x+at) - \frac{3\cdot 5}{x^2}\Gamma':(x+at) + \frac{2\cdot 3}{x}\Gamma'':(x+at) - \Gamma''':(x+at)$$
$$+ \frac{3\cdot 5}{x^3}\Delta:(x-at) - \frac{3\cdot 5}{x^2}\Delta':(x-at) + \frac{2\cdot 3}{x}\Delta'':(x-at) - \Delta''':(x-at)$$

4e CAS $n = -4$

Ou intégration de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{20}{xx}z\,.$$

23. Ici les coéficiens sont déterminés ainsi:

$$B = -A\,,\quad C = -\tfrac{3}{7}B = \tfrac{3}{7}A\,,$$
$$D = -\tfrac{2}{9}C = -\frac{2\cdot 3}{7\cdot 9}A\,,\quad E = -\frac{1}{10}D = \frac{1\cdot 2\cdot 3}{7\cdot 9\cdot 10}A$$

de sorte qu'en entiers nous aurons

$$A = 3\cdot 5\cdot 7,\ B = -3\cdot 5\cdot 7,\ C = 3\cdot 3\cdot 5,\ D = -2\cdot 5,\ E = 1$$

et l'intégrale complète sera

$$z = \frac{3\cdot 5\cdot 7}{x^4}\Gamma:(x+at) - \frac{3\cdot 5\cdot 7}{x^3}\Gamma':(x+at) + \frac{3\cdot 3\cdot 5}{x^2}\Gamma'':(x+at) - \frac{2\cdot 5}{x}\Gamma''':(x+at) + \Gamma^{IV}:(x+at)$$
$$+ \frac{3\cdot 5\cdot 7}{x^4}\Delta:(x-at) - \frac{3\cdot 5\cdot 7}{x^3}\Delta':(x-at) + \frac{3\cdot 3\cdot 5}{x^2}\Delta'':(x-at) - \frac{2\cdot 5}{x}\Delta''':(x-at) + \Delta^{IV}:(x-at)$$

5^e CAS $n=-5$

Ou intégration de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{30}{xx}\,z\,.$$

24. Les coéficiens sont déterminés de la manière suivante

$$B=-A,\quad C=-\tfrac{4}{9}B,\quad D=-\frac{3}{12}C,\quad E=-\frac{2}{14}D,\quad F=-\frac{1}{15}E$$

et prenant $A=3\cdot5\cdot7\cdot9$, nous aurons

$$A=3\cdot5\cdot7\cdot9,\quad B=-3\cdot5\cdot7\cdot9,\quad C=4\cdot3\cdot5\cdot7,\quad D=-3\cdot5\cdot7,\quad E=3\cdot5,\quad F=-1$$

et après avoir trouvé ces coéficiens, on n'a qu'à observer que l'intégrale complète sera

$$\begin{aligned} z=\frac{A}{x^5}\Gamma:(x+at)+\frac{B}{x^4}\Gamma':(x+at)+\frac{C}{x^3}\Gamma'':(x+at)+\frac{D}{x^2}\Gamma''':(x+at)+\text{etc.}\\ +\frac{A}{x^5}\varDelta:(x-at)+\frac{B}{x^4}\varDelta':(x-at)+\frac{C}{x^3}\varDelta'':(x-at)+\frac{D}{x^2}\varDelta''':(x-at)+\text{etc.}\end{aligned}$$

6^e CAS $n=-6$

Ou intégration de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{42}{xx}\,z\,.$$

25. Pour les valeurs des coéficiens on aura

$$B=-A\,,\ C=-\frac{5}{11}B,\ D=-\frac{4}{15}C,\ E=-\frac{3}{18}D\,,\ F=-\frac{2}{20}E\,,\ G=-\frac{1}{21}F$$

et prenant $A=3\cdot5\cdot7\cdot9\cdot11$, les autres seront

$$B=-3\cdot5\cdot7\cdot9\cdot11,\ C=+5\cdot3\cdot5\cdot7\cdot9,\ D=-4\cdot5\cdot7\cdot9,\ E=+2\cdot3\cdot5\cdot7,\ F=-3\cdot7,\ G=1$$

d'où l'intégrale complète devient

$$z = \frac{A}{x^6}\Gamma : (x+at) + \frac{B}{x^5}\Gamma' : (x+at) + \frac{C}{x^4}\Gamma'' : (x+at) + \frac{D}{x^3}\Gamma''' : (x+at) + \text{etc.}$$
$$+ \frac{A}{x^6}\Delta : (x-at) + \frac{B}{x^5}\Delta' : (x-at) + \frac{C}{x^4}\Delta'' : (x-at) + \frac{D}{x^3}\Delta''' : (x-at) + \text{etc.}$$

26. En général pour cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{i(i+1)}{xx}z$$

l'intégrale complète sera

$$z = \frac{A}{x^i}\Gamma : (x+at) + \frac{B}{x^{i-1}}\Gamma' : (x+at) + \frac{C}{x^{i-2}}\Gamma'' : (x+at) + \frac{D}{x^{i-3}}\Gamma''' : (x+at) + \text{etc.}$$
$$+ \frac{A}{x^i}\Delta : (x-at) + \frac{B}{x^{i-1}}\Delta' : (x-at) + \frac{C}{x^{i-2}}\Delta'' : (x-at) + \frac{D}{x^{i-3}}\Delta''' : (x-at) + \text{etc.,}$$

où les coéficiens doivent être déterminés de la manière suivante:

$$\begin{aligned}
B &= -A \\
C &= -\frac{(i-1)B}{2i-1} = +\frac{(i-1)A}{2i-1} \\
D &= -\frac{(i-2)C}{3i-3} = -\frac{(2i-4)A}{2\cdot 3(2i-1)} \\
E &= -\frac{(i-3)D}{4i-6} = +\frac{(2i-4)\,(2i-6)A}{2\cdot 3\cdot 4(2i-1)\,(2i-3)} \\
F &= -\frac{(i-4)E}{5i-10} = -\frac{(2i-6)\,(2i-8)A}{2\cdot 3\cdot 4\cdot 5(2i-1)\,(2i-3)} \\
G &= -\frac{(i-5)F}{6i-15} = +\frac{(2i-6)\,(2i-8)\,(2i-10)A}{2\cdot 3\cdot 4\cdot 5\cdot 6(2i-1)\,(2i-3)\,(2i-5)} \\
&\quad\text{etc.} \qquad\qquad\qquad \text{etc.}
\end{aligned}$$

27. Pour comprendre mieux la loi de ces coéficiens j'observe, qu'on trouve les mêmes coéficiens, en intégrant cette équation différentio-différentielle

$$\frac{xxddy}{dx^2} + \frac{2xxdy}{dx} - i(i+1)y = 0,$$

si l'on suppose l'intégrale exprimée par cette série

$$y=\frac{A}{x^i}+\frac{B}{x^{i-1}}+\frac{C}{x^{i-2}}+\frac{D}{x^{i-3}}+\text{ etc.}$$

Or si i est un nombre entier, cette même équation nous fournit les coéficiens en ordre retrograde, car si nous supposons

$$y=\mathrm{A}+\frac{\mathrm{B}}{x}+\frac{\mathrm{C}}{x^2}+\frac{\mathrm{D}}{x^3}+\frac{\mathrm{E}}{x^4}+\frac{\mathrm{F}}{x^5}+\text{ etc.},$$

nous trouvons:

$$\mathrm{B}=-\frac{i(i+1)}{2}\mathrm{A}$$

$$\mathrm{C}=-\frac{(i-1)\;(i+2)}{4}\mathrm{B}=+\frac{i(ii-1)\;(i+2)}{2\cdot 4}\mathrm{A}$$

$$\mathrm{D}=-\frac{(i-2)\;(i+3)}{6}\mathrm{C}=-\frac{i(ii-1)\;(ii-4)\;(i+3)}{2\cdot 4\cdot 6}\mathrm{A}$$

$$\mathrm{E}=-\frac{(i-3)\;(i+4)}{8}\mathrm{D}=+\frac{i(ii-1)\;(ii-4)\;(ii-9)\;(i+4)}{2\cdot 4\cdot 6\cdot 8}\mathrm{A}$$

$$\mathrm{F}=-\frac{(i-4)\;(i+5)}{10}\mathrm{E}=-\frac{i(ii-1)\;(ii-4)\;(ii-9)\;(ii-16)\;(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10}\mathrm{A}$$

etc. etc.

28. Donc si nous marquons les fonctions en ascendant par intégration, de cette manière

$$'\Gamma: s=\int ds\,\Gamma: s,\quad ''\Gamma: s=\int ds\,'\Gamma: s,\quad '''\Gamma: s=\int ds\,''\Gamma: s\ \text{ etc.}$$

l'intégrale de notre équation $\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{i(i+1)}{xx}z$ sera

$$z=\mathrm{A}\Gamma:(x+at)+\frac{\mathrm{B}}{x}\,'\Gamma:(x+at)+\frac{\mathrm{C}}{xx}\,''\Gamma:(x+at)+\frac{\mathrm{D}}{x^3}\,'''\Gamma:(x+at)+\text{ etc.}$$

$$+\mathrm{A}\Delta:(x-at)+\frac{\mathrm{B}}{x}\,'\Delta:(x-at)+\frac{\mathrm{C}}{xx}\,''\Delta:(x-at)+\frac{\mathrm{D}}{x^3}\,'''\Delta:(x-at)+\text{ etc.}$$

et pour les diverses valeurs de i ces coéficiens seront

i	A	$-$B	$+$C	$-$D	$+$E	$-$F
1	1	$\frac{1\cdot2}{2}$				
2	1	$\frac{2\cdot3}{2}$	$\frac{2\cdot3\cdot4}{2\cdot4}$			
3	1	$\frac{3\cdot4}{2}$	$\frac{3\cdot8\cdot5}{2\cdot4}$	$\frac{3\cdot8\cdot5\cdot6}{2\cdot4\cdot6}$		
4	1	$\frac{4\cdot5}{2}$	$\frac{4\cdot15\cdot6}{2\cdot4}$	$\frac{4\cdot15\cdot12\cdot7}{2\cdot4\cdot6}$	$\frac{4\cdot15\cdot12\cdot7\cdot8}{2\cdot4\cdot6\cdot8}$	
5	1	$\frac{5\cdot6}{2}$	$\frac{5\cdot24\cdot7}{2\cdot4}$	$\frac{5\cdot24\cdot21\cdot8}{2\cdot4\cdot6}$	$\frac{5\cdot24\cdot21\cdot16\cdot9}{2\cdot4\cdot6\cdot8}$	$\frac{5\cdot24\cdot21\cdot16\cdot9\cdot10}{2\cdot4\cdot6\cdot8\cdot10}$
6	1	$\frac{6\cdot7}{2}$	$\frac{6\cdot35\cdot8}{2\cdot4}$	$\frac{6\cdot35\cdot32\cdot9}{2\cdot4\cdot6}$	$\frac{6\cdot35\cdot32\cdot27\cdot10}{2\cdot4\cdot6\cdot8}$	$\frac{6\cdot35\cdot32\cdot27\cdot20\cdot11}{2\cdot4\cdot6\cdot8\cdot10}$
7	1	$\frac{7\cdot8}{2}$	$\frac{7\cdot48\cdot9}{2\cdot4}$	$\frac{7\cdot48\cdot45\cdot10}{2\cdot4\cdot6}$	$\frac{7\cdot48\cdot45\cdot40\cdot11}{2\cdot4\cdot6\cdot8}$	$\frac{7\cdot48\cdot45\cdot40\cdot33\cdot12}{2\cdot4\cdot6\cdot8\cdot10}$

etc. etc.

29. J'observe encore, que l'intégrale de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{n(n+1)}{xx}z$$

se peut exprimer d'une double manière, l'une finie, et l'autre infinie, puisque on peut mettre tant $i=n$ que $i=-n-1$. Pour rendre l'une et l'autre plus évidente, posons

$$x+at=p \quad \text{et} \quad x-at=q,$$

et que P marque une fonction quelconque de p et Q de q; cela posé, la première forme sera

$$z=\frac{A}{x^n}(P+Q)+\frac{B}{x^{n-1}}\left(\frac{dP}{dp}+\frac{dQ}{dq}\right)+\frac{C}{x^{n-2}}\left(\frac{ddP}{dp^2}+\frac{ddQ}{dq^2}\right)+\text{etc.},$$

les coéficiens étant déterminés ainsi:

$$B=-A\,,\quad C=-\frac{2(n-1)}{2(2n-1)}B\,,\quad D=-\frac{2(n-2)}{3(2n-2)}C\,,$$

$$E=-\frac{2(n-3)}{4(2n-3)}D\,,\quad F=-\frac{2(n-4)}{5(2n-4)}E \text{ etc.}$$

Or l'autre forme est

$$z = Ax^{n+1}(P+Q) + Bx^{n+2}\left(\frac{dP}{dp}+\frac{dQ}{dq}\right) + Cx^{n+3}\left(\frac{ddP}{dp^2}+\frac{ddQ}{dq^2}\right) + \text{ etc.},$$

où les coéficiens ont les valeurs suivantes:

$$B = -A\,,\quad C = -\frac{2(n+2)}{2(2n+3)}B\,,\quad D = -\frac{2(n+3)}{3(2n+4)}C\,,$$

$$E = -\frac{2(n+4)}{4(2n+5)}D\,,\quad F = -\frac{2(n+5)}{5(2n+6)}E \text{ etc.}$$

Il est bien remarquable que ces deux expressions sont équivalentes, et que dans les cas $n=0$, et $n=-1$, l'une et l'autre devient infinie, et se réduit à $z = P + Q$.

30. Puisque cette réduction ne paroit pas d'abord, il est bon de la démontrer. Soit donc $n = 0$, et pour la première forme on aura

$$A=1\,,\quad B=-1\,,\quad C=\frac{2}{2},\quad D=-\frac{2^2}{2\cdot 3}\,,\quad E=+\frac{2^3}{2\cdot3\cdot4}\,,\quad F=-\frac{2^4}{2\cdot3\cdot4\cdot5} \text{ etc.},$$

d'où nous tirons

$$z = 1(P+Q) - \frac{x}{1}\left(\frac{dP}{dp}+\frac{dQ}{dq}\right) + \frac{2x^2}{1\cdot2}\left(\frac{ddP}{dp^2}+\frac{ddQ}{dq^2}\right) - \frac{2^2x^3}{1\cdot2\cdot3}\left(\frac{d^3P}{dp^3}+\frac{d^3Q}{dq^3}\right) + \text{ etc.}$$

Puisque chaque membre contient deux termes, soit M la série qui contient les termes dépendans de P et N celle qui contient Q, de sorte que $z = M + N$; maintenant puisque $P = \Gamma\colon p = \Gamma\colon (x + at)$, à cause de $p = x + at$, si nous posons $p - 2x$ au lieu de x, on aura par la nature des différentiels

$$\Gamma\colon(p-2x) = P - \frac{2x}{1}\times\frac{dP}{dp} + \frac{2^2x^2}{1\cdot2}\times\frac{ddP}{dp^2} - \frac{2^3x^3}{1\cdot2\cdot3}\times\frac{d^3P}{dp^3} + \frac{2^4x^4}{1\cdot2\cdot3\cdot4}\times\frac{d^4P}{dp^4} - \text{ etc.}$$

donc, puisque

$$M = P - \frac{x}{1}\times\frac{dP}{dp} + \frac{2x^2}{1\cdot2}\times\frac{ddP}{dp^2} - \frac{2^2x^3}{1\cdot2\cdot3}\times\frac{d^3P}{dp^3} + \text{ etc.},$$

il est évident, que

$$2M - \Gamma\colon(p-2x) = P = \Gamma\colon p,$$

et partant

$$M = \tfrac{1}{2}\Gamma : p + \tfrac{1}{2}\Gamma : (p - 2x) = \tfrac{1}{2}\Gamma : (x + at) + \tfrac{1}{2}\Gamma : (at - x)$$

De la même manière on verra que

$$N = \tfrac{1}{2}\Delta : q + \tfrac{1}{2}\Delta : (q - 2x) = \tfrac{1}{2}\Delta : (x - at) + \tfrac{1}{2}\Delta : (-at - x),$$

et par conséquent $z = M + N$ se trouve égale à la somme de deux fonctions, l'une de $x + at$, et l'autre de $x - at$.

31. L'autre forme, posant $n = 0$, donne

$$A = 1,\ B = -1,\ C = \frac{2}{3},\ D = -\frac{2^2}{3\cdot 4},\ E = +\frac{2^3}{3\cdot 4\cdot 5},\ F = -\frac{2^4}{3\cdot 4\cdot 5\cdot 6}\ \text{etc.}$$

d'où l'on obtient

$$z = x(P+Q) - \frac{2x^2}{2\cdot 3}\left(\frac{dP}{dp} + \frac{dQ}{dq}\right) + \frac{2^2x^3}{1\cdot 2\cdot 3}\left(\frac{ddP}{dp^2} + \frac{ddQ}{dq^2}\right) - \frac{2^3x^4}{1\cdot 2\cdot 3\cdot 4}\left(\frac{d^3P}{dp^3} + \frac{d^3Q}{dq^3}\right) + \text{etc.}$$

Posons

$$z = M + N,$$

de sorte que M contienne les termes renfermans P et N les Q. Considérons à présent la forme

$$\int P dp = {}'\Gamma : p = {}'\Gamma : (x + at),$$

et posons $p - 2x$ au lieu de p, pour avoir

$${}'\Gamma : (p - 2x) = {}'\Gamma : p - \frac{2x}{1} \times P + \frac{2^2x^2}{1\cdot 2} \times \frac{dP}{dp} - \frac{2^3x^3}{1\cdot 2\cdot 3} \times \frac{ddP}{dp^2} + \text{etc.}$$

Or

$$M = \frac{x}{1}P - \frac{2x^2}{1\cdot 2} \times \frac{dP}{dp} + \frac{2^2x^3}{1\cdot 2\cdot 3} \times \frac{ddP}{dp^2} - \text{etc.},$$

donc

$$2M + {}'\Gamma : (p - 2x) = {}'\Gamma : p$$

et partant

$$M = \tfrac{1}{2}{}'\Gamma : p - \tfrac{1}{2}{}'\Gamma : (p - 2x) = \tfrac{1}{2}{}'\Gamma : (x + at) - \tfrac{1}{2}{}'\Gamma : (at - x)$$

et de la même manière on trouve

$$N = \tfrac{1}{2}{}'\Delta : q - \tfrac{1}{2}{}'\Delta : (q - 2x) = \tfrac{1}{2}{}'\Delta : (x - at) - \tfrac{1}{2}{}'\Delta : (-at - x)$$

d'où il est clair que $z = M + N$ devient comme ci-devant la somme des deux fonctions, l'une de $x + at$, et l'autre de $x - at$. Cette réduction pour le cas le plus simple est très remarquable.

32. Par les mêmes principes on peut prouver l'identité des deux expressions pour les autres cas. Soit $n = 1$, et la première forme donnant $A = 1$ et $B = -1$, l'intégrale est

$$z = \frac{1}{x}(P+Q) - 1\left(\frac{dP}{dp} + \frac{dQ}{dq}\right).$$

Or l'autre forme donne ces coéficiens

$$B = -A, \quad C = +\frac{3}{5}A, \quad D = -\frac{2\cdot 4}{5\cdot 6}A, \quad E = +\frac{2^2\cdot 5}{5\cdot 6\cdot 7}A, \quad F = -\frac{2^3\cdot 6}{5\cdot 6\cdot 7\cdot 8}A \quad \text{etc.}$$

Posons $A = \frac{2^3}{1\cdot 2\cdot 3\cdot 4}$, pour avoir la forme suivante de l'intégrale, dans laquelle je ne mets que les termes renfermant $P = \Gamma\colon (x + at)$, puisque la réduction pour les Q est la même:

$$z = \frac{2^3x^2}{1\cdot\cdot 4}P - \frac{2^3x^3}{1\cdot\cdot 4}\cdot\frac{dP}{dp} + \frac{2^3\cdot 3x^4}{1\cdot\cdot\cdot 5}\cdot\frac{ddP}{dp^2} - \frac{2^4\cdot 4x^5}{1\cdot\cdot\cdot 6}\cdot\frac{d^3P}{dp^3} + \frac{2^5\cdot 5x^6}{1\cdot\cdot\cdot 7}\cdot\frac{d^4P}{dp^4}\ \text{etc.}$$

que je décompose en deux parties, savoir

$$\begin{aligned} z = {} & \frac{2x^2}{1\cdot 2}P - \frac{2^2\cdot x^3}{1\cdot 2\cdot 3}\cdot\frac{dP}{dp} + \frac{2^3x^4}{1\cdot\cdot 4}\cdot\frac{ddP}{dp^2} - \frac{2^4x^5}{1\cdot\cdot\cdot 5}\cdot\frac{d^3P}{dp^3} + \frac{2^5x^6}{1\cdot\cdot\cdot 6}\cdot\frac{d^4P}{dp^4} - \text{etc.} \\ & - \frac{2^2x^2}{1\cdot 2\cdot 3}P + \frac{2^3x^3}{1\cdot\cdot 4}\cdot\frac{dP}{dp} - \frac{2^4x^4}{1\cdot\cdot 5}\cdot\frac{ddP}{dp^2} + \frac{2^5x^5}{1\cdot\cdot\cdot 6}\cdot\frac{d^3P}{dp^3} - \frac{2^6x^6}{1\cdot\cdot\cdot 7}\cdot\frac{d^4P}{dp^4} + \text{etc.} \end{aligned}$$

pour avoir une telle forme

$$z = M - N,$$

où M et N marquent les séries trouvées, qui étant assès régulières, la valeur de l'une et de l'autre se découvrira ainsi.

33. Pour la première série M, je pose

$$P = \frac{ddR}{dp^2},$$

où R est aussi une fonction de $x + at$, et j'aurai

$$M = \frac{2x^2}{1\cdot 2}\cdot\frac{ddR}{dp^2} - \frac{2^2x^3}{1\cdot 2\cdot 3}\cdot\frac{d^3R}{dp^3} + \frac{2^3x^4}{1\cdot\cdot 4}\cdot\frac{d^4R}{dp^4} - \text{etc.}$$

Mais posant

$$R = \Gamma : p,$$

il me vient

$$\Gamma : (p - 2x) = R - \frac{2x}{1}\cdot\frac{dR}{dp} + \frac{2^2x^2}{1\cdot 2}\cdot\frac{ddR}{dp^2} - \frac{2^3x^3}{1\cdot 2\cdot 3}\cdot\frac{d^3R}{dp^3} + \text{ etc.}$$

et partant

$$\Gamma : (p - 2x) = R - \frac{2x}{1}\cdot\frac{dR}{dp} + 2M,$$

d'où l'on tire la valeur de M, savoir

$$M = \tfrac{1}{2}\Gamma : (p - 2x) - \tfrac{1}{2}\Gamma : p + x\Gamma' : p.$$

Pour l'autre série je fais

$$P = \frac{d^3S}{dp^3}$$

pour avoir

$$N = \frac{2^2x^2}{1\cdot 2\cdot 3}\cdot\frac{d^3S}{dp^3} - \frac{2^3x^3}{1\cdot\cdot 4}\cdot\frac{d^4S}{dp^4} + \frac{2^4x^4}{1\cdot\cdot\cdot 5}\cdot\frac{d^5S}{dp^5} - \text{ etc.}$$

Mais puisque

$$\frac{d^3S}{dp^3} = \frac{ddR}{dp^2},$$

il suit que

$$\frac{dS}{dp} = R = \Gamma : p,$$

donc

$$S = {}'\Gamma : p$$

et

$${}'\Gamma : (p - 2x) = S - \frac{2x}{1}\cdot\frac{dS}{dp} + \frac{2^2x^2}{1\cdot 2}\cdot\frac{ddS}{dp^2} - \frac{2^3x^3}{1\cdot 2\cdot 3}\cdot\frac{d^3S}{dp^3} + \frac{2^4x^4}{1\cdot\cdot 4}\cdot\frac{d^4S}{dp^4} - \text{ etc.}$$

et partant

$${}'\Gamma : (p - 2x) = {}'\Gamma : p - 2x\Gamma : p + 2xx\Gamma' : p - 2xN,$$

de sorte que

$$N = -\frac{1}{2x}{}'\Gamma : (p - 2x) + \frac{1}{2x}{}'\Gamma : p - \Gamma : p + x\Gamma' : p,$$

par conséquent

$$M-N=\frac{1}{2}\Gamma:(p-2x)+\frac{1}{2x}\,'\Gamma:(p-2x)+\frac{1}{2}\Gamma:p-\frac{1}{2x}\,'\Gamma:p$$

ce qui est la valeur de z.

34. Puisque $p=x+at$, en doublant les termes, et posant Γ au lieu de $'\Gamma$, la partie de l'intégrale, qui renferme la lettre P, se réduit à cette forme

$$z=\Gamma':(x+at)-\frac{1}{x}\Gamma:(x+at)+\Gamma':(-x+at)+\frac{1}{x}\Gamma:(-x+at),$$

or la même méthode nous fournit pour les termes Q

$$z=\Delta':(x-at)-\frac{1}{x}\Delta:(x-at)+\Delta':(-x-at)+\frac{1}{x}\Delta:(-x-at)$$

et ces deux expressions combinées ensemble donnent l'intégrale trouvée par la première forme.

Soit pour cela

$$\Delta:(-x-at)-\Gamma:(x+at)=P=\text{fonct.}\ (x+at=p),$$

et la différentiation donnera

$$-\Delta':(-x-at)-\Gamma':(x+at)=\frac{dP}{dp};$$

posant de même

$$\Gamma:(-x+at)-\Delta:(x-at)=Q=\text{fonct.}\ (x-at=q),$$

on aura par la différentiation

$$-\Gamma':(-x+at)-\Delta':(x-at)=\frac{dQ}{dq},$$

et de là il est clair qu'on aura, comme ci-dessus

$$z=\frac{1}{x}(P+Q)-1\left(\frac{dP}{dp}+\frac{dQ}{dq}\right).$$

35. Ces réductions que je viens de faire pour les cas $n=0$ et $n=1$, peuvent être appliquées avec le même succès à des plus grands nombres pris pour n, mais il est aisé de prévoir, qu'elles deviendront de plus en plus embarrassantes. Cependant il suffit d'avoir fait voir la possibilité de réduire la forme infinie de l'intégrale à la finie, dans les cas, où n est un nombre entier. Mais pour

les cas, où n est une fraction, les deux formes trouvées pour l'intégrale dans le § 29 deviennent infinies, et il ne paroit aucun moyen de réduire l'une ou l'autre à une forme finie. On y rencontre les mêmes difficultés que dans les cas irrésolubles de l'équation de Riccati, et quoique j'aie trouvé[1]) moyen de construire ces cas par une méthode tout à fait particulière, il ne semble point, que cette méthode puisse être appliquée aux cas dont il s'agit ici. Mais non obstant le nombre infini des termes, dont les intégrales de ces cas sont composées, elles ne sont pas moins complètes, puisqu'elles renferment deux fonctions absolument indéterminées et arbitraires.

36. Or pour trouver plus facilement les intégrales complètes de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{n(n+1)}{xx}z$$

qu'on pose

$$x+at=p \quad \text{et} \quad x-at=q,$$

ensuite soit P une fonction quelconque de p et Q une fonction quelconque de q, qu'on fasse outre cela pour abreger:

$$\begin{aligned} &P'=\int Pdp, \quad P''=\int P'dp, \quad P'''=\int P''dp, \quad P^{IV}=\int P'''dp \quad \text{etc.}\\ &Q'=\int Qdq, \quad Q''=\int Q'dq, \quad Q'''=\int Q''dq, \quad Q^{IV}=\int Q'''dq \quad \text{etc.}\end{aligned}$$

toutes ces valeurs pouvant aisément être représentées par des quadratures de lignes courbes, enfin qu'on forme les coéficiens suivans:

$$\begin{aligned} \mathrm{A}&=+1\\ \mathrm{B}&=+\frac{n(n+1)}{2}\\ \mathrm{C}&=+\frac{(n-1)n(n+1)\,(n+2)}{2\cdot 4}\\ \mathrm{D}&=+\frac{(n-2)\,(n-1)\,n(n+1)\,(n+2)\,(n+3)}{2\cdot 4\cdot 6}\\ \mathrm{E}&=+\frac{(n-3)\,(n-2)\cdots\cdots\cdots\cdots(n+4)}{2\cdot 4\cdot 6\cdot 8}\\ \mathrm{F}&=+\frac{(n-4)\,(n-3)\cdots\cdots\cdots\cdots(n+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10}\\ &\text{etc.}\end{aligned}$$

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1) L. EULER: Mémoires 11 et 31. *Constructio aequationum quarundam differentialium.* Nova acta erud. 1733, p. 369. *Constructio aequationis differentialis* $ax^n dx = dy + y^2 dx$. Comment. acad. sc. Petrop. 6, 1738, p. 124. *LEONHARDI EULERI Opera omnia,* series I, vol. 22, p. 15 et 19. H. D.

et l'intégrale cherchée sera

$$z = \mathrm{A}(P+Q) - \frac{\mathrm{B}}{x}(P'+Q') + \frac{\mathrm{C}}{x^2}(P''+Q'') - \frac{\mathrm{D}}{x^3}(P'''+Q''') + \text{ etc.}$$

37. Aiant trouvé cette intégrale ou valeur de z, on aura aussi les intégrales de toutes les équations qui en sont transformées, comme posant $z = x^{-m}u$, à cause de

$$\left(\frac{ddz}{dt^2}\right) = x^{-m}\left(\frac{ddu}{dt^2}\right) \quad \text{et}$$

$$\left(\frac{dz}{dx}\right) = x^{-m}\left(\frac{du}{dx}\right) - mx^{-m-1}u$$

$$\left(\frac{ddz}{dx^2}\right) = x^{-m}\left(\frac{ddu}{dx^2}\right) - 2mx^{-m-1}\left(\frac{du}{dx}\right) + m(m+1)x^{-m-2}u$$

l'équation transformée sera

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) - \frac{2m}{x}\left(\frac{du}{dx}\right) + \frac{(m-n)\ (m+n+1)}{xx}u$$

et partant son intégrale sera

$$u = \mathrm{A}x^m(P+Q) - \mathrm{B}x^{m-1}(P'+Q') + \mathrm{C}x^{m-2}(P''+Q'') - \mathrm{D}x^{m-3}(P'''+Q''') + \text{ etc.}$$

38. Développons les cas les plus simples de cette équation généralisée, en conservant les mêmes significations

$$p = x + at \quad \text{et} \quad q = x - at,$$

et des fonctions qui en sont formées P, P', P'', P''' etc., Q, Q', Q'', Q''' etc.

1°
$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) - \frac{2m}{x}\left(\frac{du}{dx}\right) + \frac{m(m+1)}{xx}u$$

l'intégrale est $u = x^m(P+Q)$.

2°
$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{ddu}{dx^2}\right) - \frac{2m}{x}\left(\frac{du}{dx}\right) + \frac{(m-1)\ (m+2)}{xx}u$$

l'intégrale $u = x^m(P+Q) - \frac{1\cdot 2}{2}x^{m-1}(P'+Q')$.

3°
$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)=\left(\frac{ddu}{dx^2}\right)-\frac{2m}{x}\left(\frac{du}{dx}\right)+\frac{(m-2)\,(m+3)}{xx}u$$

dont l'intégrale est

$$u=x^m(P+Q)-\frac{2\cdot 3}{2}x^{m-1}(P'+Q')+\frac{1\cdot 2\cdot 3\cdot 4}{2\cdot 4}x^{m-2}(P''+Q'')\,.$$

4°
$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)=\left(\frac{ddu}{dx^2}\right)-\frac{2m}{x}\left(\frac{du}{dx}\right)+\frac{(m-3)\,(m+4)}{xx}u$$

dont l'intégrale est

$$u=x^m(P+Q)-\frac{3\cdot 4}{2}x^{m-1}(P'+Q')+\frac{2\cdot 3\cdot 4\cdot 5}{2\cdot 4}x^{m-2}(P''+Q'')$$
$$-\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{2\cdot 4\cdot 6}x^{m-3}(P'''+Q''')\,.$$

5°
$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)=\left(\frac{ddu}{dx^2}\right)-\frac{2m}{x}\left(\frac{du}{dx}\right)+\frac{(m-4)\,(m+5)}{xx}u$$

dont l'intégrale

$$u=x^m(P+Q)-\frac{4\cdot 5}{2}x^{m-1}(P'+Q')+\frac{3\cdot 4\cdot 5\cdot 6}{2\cdot 4}x^{m-2}(P''+Q'')$$
$$-\frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}{2\cdot 4\cdot 6}x^{m-3}(P'''+Q''')$$
$$+\frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 6\cdot 8}x^{m-4}(P^{IV}+Q^{IV})\,.$$

6°
$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)=\left(\frac{ddu}{dx^2}\right)-\frac{2m}{x}\left(\frac{du}{dx}\right)+\frac{(m-5)\,(m+6)}{xx}u$$

dont l'intégrale

$$u=x^m(P+Q)-\frac{5\cdot 6}{2}x^{m-1}(P'+Q')+\frac{4\cdot 5\cdot 6\cdot 7}{2\cdot 4}x^{m-2}(P''+Q'')-\frac{3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}{2\cdot 4\cdot 6}x^{m-3}(P'''+Q''')$$
$$+\frac{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9}{2\cdot 4\cdot 6\cdot 8}x^{m-4}(P^{IV}+Q^{IV})-\frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\cdot 9\cdot 10}{2\cdot 4\cdot 6\cdot 8\cdot 10}x^{m-5}(P^{V}+Q^{V})\,.$$

39. Jusqu'ici je n'ai donné que les intégrales pour les cas de l'équation proposée qui en sont susceptibles, sans y avoir été conduit par une méthode certaine et directe, car toute l'analyse dont je me suis servi est uniquement fondée sur une heureuse conjecture, qui regarde la forme de ces intégrales, et que le pur hazard m'a quasi fournie. Il est donc d'autant plus important de

découvrir une méthode directe, qui nous puisse mener au même but, qu'il n'y a aucun doute, qu'une telle méthode n'apporte de très grands éclaircissemens dans cette nouvelle carrière; or la considération de ces intégrales nous porte aisément à penser, qu'il doit y avoir une méthode, qui, moyennant une certaine transformation conduise des cas plus simples aux plus compliqués, de la même manière que dans le développement de la fameuse équation de Riccati on a réussi à déduire les cas plus difficiles des plus simples, en y employant une certaine substitution. C'est donc par une semblable méthode, que je m'en vais enseigner la résolution de l'équation, que j'ai traitée jusqu'ici.

40. L'équation proposée étant donc

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{k}{xx}z$$

je la transforme par le moyen de cette substitution

$$u=\left(\frac{dz}{dx}\right)+\frac{m}{x}z$$

en cherchant une équation, qui détermine la fonction u par t et x; or il est aisé de prévoir, que l'équation transformée aura une telle forme

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)=\left(\frac{ddu}{dx^2}\right)-\frac{k'}{xx}u$$

semblable à la proposée, avec cette seule différence que le nombre k' sera différent du nombre k; ou bien sans supposer une telle prévoyance, on peut regarder cette dernière équation comme la proposée, et introduire z à la place de u par la substitution indiquée.

41. Supposant donc

$$u=\left(\frac{dz}{dx}\right)+\frac{m}{x}z\,,$$

on en tire d'abord

$$\left(\frac{ddu}{dt^2}\right)=\left(\frac{d^3z}{dt^2dx}\right)+\frac{m}{x}\left(\frac{ddz}{dt^2}\right).$$

Or la première équation donnant

$$\left(\frac{ddz}{dt^2}\right) = aa\left(\frac{ddz}{dx^2}\right) - \frac{aak}{xx}z,$$

si nous différentions par la seule variabilité de x, nous aurons

$$\left(\frac{d^3z}{dt^2dx}\right) = aa\left(\frac{d^3z}{dx^3}\right) - \frac{aak}{xx}\left(\frac{dz}{dx}\right) + \frac{2aak}{x^3}z$$

d'où nous tirons

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right) = \left(\frac{d^3z}{dx^3}\right) + \frac{m}{x}\left(\frac{ddz}{dx^2}\right) - \frac{k}{xx}\left(\frac{dz}{dx}\right) - \frac{(m-2)k}{x^3}z.$$

Maintenant la même substitution donne

$$\left(\frac{du}{dx}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) - \frac{m}{xx}z$$

et de là

$$\left(\frac{ddu}{dx^2}\right) = \left(\frac{d^3z}{dx^3}\right) + \frac{m}{x}\left(\frac{ddz}{dx^2}\right) - \frac{2m}{xx}\left(\frac{dz}{dx}\right) + \frac{2m}{x^3}z,$$

or

$$-\frac{k'}{xx}u = -\frac{k'}{xx}\left(\frac{dz}{dx}\right) - \frac{mk'}{x^3}z$$

et partant ces deux dernières expressions ensemble doivent être égales à celle, qui vient d'être trouvée pour $\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)$.

42. Puisque les termes $\left(\frac{d^3z}{dx^3}\right)$ et $\left(\frac{ddz}{dx^2}\right)$ se détruisent d'eux mêmes, il ne reste qu'à égaler séparément ceux qui sont affectés par $\left(\frac{dz}{dx}\right)$ et par z, d'où nous tirons ces deux égalités :

$$-k = -2m - k' \quad \text{et} \quad -(m-2)k = 2m - mk',$$

ou

$$m = \frac{k-k'}{2} \quad \text{et} \quad 2k = m(k - k' + 2),$$

donc

$$(k-k')^2 - 2k - 2k' = 0,$$

et partant

$$k' = k + 1 \pm \sqrt{(1+4k)} \quad \text{et} \quad m = \frac{-1 \mp \sqrt{(1+4k)}}{2}.$$

Par conséquent si nous avons réussi à trouver l'intégrale de cette équation:

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{k}{xx}z,$$

nous aurons aussi l'intégrale de celle-ci:

$$\frac{1}{aa}\left(\frac{ddu}{dt^2}\right)=\left(\frac{ddu}{dx^2}\right)-\frac{k'}{xx}u$$

posant $k'=k+1\pm\sqrt{(1+4k)}$, puisque $u=\left(\frac{dz}{dx}\right)+\frac{k-k'}{2x}z$.

43. Or aiant assigné ci-dessus, indépendemment de ces recherches, l'intégrale de notre équation pour le cas $k=0$, cette nouvelle méthode nous fournit l'intégrale pour le cas $k'=2$, et supposant ensuite $k=2$, nous en tirons le troisième cas $k'=3+\sqrt{9}=6$, et celui-ci, posant $k=6$, nous conduit au quatrième $k'=7+\sqrt{25}=12$, lequel, en faisant $k=12$, nous mène au cinquième $k'=13+\sqrt{49}=20$, et ainsi de suite selon cette progression:

k	0	2	6	12	20	30	42	etc.
k'	2	6	12	20	30	42	56	etc.

d'où l'on voit que toutes les valeurs de k sont comprises dans cette formule générale $k=i(i+1)$, et ce cas considéré en général nous conduit au suivant

$$k'=i(i+1)+1+\sqrt{[1+4i(i+1)]}=ii+3i+2=(i+1)(i+2),$$

d'où nous tirons tous les cas dont j'ai assigné ci-dessus les intégrales.

44. Donc si nous connoissons pour le cas $k=n(n+1)$ l'intégrale de cette équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{n(n+1)}{xx}z,$$

qui soit $z=V$, nous en tirerons, pour le cas $k'=(n+1)(n+2)$ l'intégrale de celle-ci:

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{(n+1)(n+2)}{xx}z,$$

qui sera

$$z = \left(\frac{dV}{dx}\right) - \frac{(n+1)}{x} V,$$

ou bien, puisqu'on peut multiplier la valeur de z par une constante quelconque, cette dernière intégrale peut être exprimée ainsi:

$$z = \frac{1}{x} V - \frac{1}{n+1}\left(\frac{dV}{dx}\right).$$

Par conséquent, puisque pour le cas $k = 0$, l'intégrale est

$$z = \Gamma : (x + at) + \Delta : (x - at),$$

en écrivant S pour cette double fonction, nous aurons pour l'équation

$$\frac{1}{aa}\left(\frac{ddz}{dt^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{k}{xx} z$$

es intégrales suivantes:

si	
$k = 0$	$z = S$
$k = 1 \cdot 2$	$z = \frac{1}{x} S - \left(\frac{dS}{dx}\right)$
$k = 2 \cdot 3$	$z = \frac{3}{2x^2} S - \frac{3}{2x}\left(\frac{dS}{dx}\right) + \frac{1}{2}\left(\frac{ddS}{dx^2}\right)$
$k = 3 \cdot 4$	$z = \frac{5}{2x^3} S - \frac{5}{2x^2}\left(\frac{dS}{dx}\right) + \frac{1}{x}\left(\frac{ddS}{dx^2}\right) - \frac{1}{2 \cdot 3}\left(\frac{d^3S}{dx^3}\right)$
etc.	etc.

45. Soit, pour le cas $k = n(n+1)$, l'intégrale

$$z = \frac{A}{x^n} S - \frac{B}{x^{n-1}}\left(\frac{dS}{dx}\right) + \frac{C}{x^{n-2}}\left(\frac{ddS}{dx^2}\right) - \frac{D}{x^{n-3}}\left(\frac{d^3S}{dx^3}\right) + \frac{E}{x^{n-4}}\left(\frac{d^4S}{dx^4}\right) - \text{etc.}$$

et pour le cas $k = (n+1)(n+2)$, à cause de

$$z = \left(\frac{dV}{dx}\right) - \frac{(n+1)}{x} V,$$

nous aurons:

$$\left(\frac{dV}{dx}\right) = -\frac{nA}{x^{n+1}}S + \frac{A}{x^n}\left(\frac{dS}{dx}\right) - \frac{B}{x^{n-1}}\left(\frac{ddS}{dx^2}\right) + \frac{C}{x^{n-2}}\left(\frac{d^3S}{dx^3}\right) - \text{etc.}$$
$$+ \frac{(n-1)B}{x^n} - \frac{(n-2)C}{x^{n-1}} + \frac{(n-3)D}{x^{n-2}} - \text{etc.}$$

$$-\frac{(n+1)}{x}V = -\frac{(n+1)A}{x^{n+1}}S + \frac{(n+1)B}{x^n}\left(\frac{dS}{dx}\right) - \frac{(n+1)C}{x^{n-1}}\left(\frac{ddS}{dx^2}\right) + \frac{(n+1)D}{x^{n-2}}\left(\frac{d^3S}{dx^3}\right) - \text{etc.}$$

et partant l'intégrale pour le cas $k = (n+1)(n+2)$ en changeant les signes sera

$$z = \frac{(2n+1)A}{x^{n+1}}S - \frac{(A+2nB)}{x^n}\left(\frac{dS}{dx}\right) + \frac{(B+[2n-1]C)}{x^{n-1}}\left(\frac{ddS}{dx^2}\right) - \frac{(C+[2n-2]D)}{x^{n-2}}\left(\frac{d^3S}{dx^3}\right) + \text{etc.}$$

et il est évident que cette méthode nous fournit les mêmes intégrales que la précédente; quoique les valeurs pour chaque nombre n ne paroissent point si ouvertement, cependant une légère attention nous découvrira bientôt la même loi de progression entre les coéficiens A, B, C, D etc., qui a été trouvée ci-dessus.

DE USU FUNCTIONUM DISCONTINUARUM IN ANALYSI

Commentatio 322 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 11 (1765), 1767, p. 67—102
Summarium ibidem p. 5—7

SUMMARIUM

Qui problematis de motu cordarum vibratorio solutiones dederunt Geometrae, non nisi illum casum contemplati sunt, quo figura cordae, ab initio motus impressa, regularis et certa quadam aequatione comprehensa esse supponitur; alterum vero casum, si haec figura fuerit discontinua sive irregularis, negarunt ad Analysin pertinere aut motus inde secuturos posse ulla ratione definiri. Quae quidem investigatio cum non tantum ex Analysi minime proscribenda, sed ad eam potius novis insignibus subsidiis ditandam imprimis apta videatur, iam pridem Illustris huius dissertationis Auctor motum cordarum vibrantium generalissime ita definivit, ut ea solutio ad omnes motus et figuras, cordae in statu initiali impressas, pateret. Statim vero perspexit Illustris Vir, problematis difficultates superare Analyseos adhuc excultas vires novamque ad id solvendum calculi integralis partem requiri, cuius non complura modo in his Commentariis specimina dedit, sed plenam quoque eius tractationem absoluto Operi de Calculo Integrali, quod typis nunc hic exscribitur, inseruit.

Concipiatur scilicet corda aliqua tensa, eique arbitraria quaevis figura et in singulis simul ipsius punctis arbitraria imprimatur celeritas. Ductis igitur coordinatis x et y, evidens est, applicatam non pro diversis solum ipsius curvae punctis, sed etiam in eodem curvae puncto pro singulis temporum momentis variare adeoque y fore functionem duarum variabilium x et t simul, quarum neutra per alteram determinatur. Ut porro in aequatione pro cordae huius motu inventa status initialis exprimatur, posito $t = 0$ relatio inter x et y inde orta figuram cordae initialem repraesentare debebit; quae cum libero manus ductu formata supponatur, functio quaedam discontinua aequationem ingredietur, atque similiter, ut secundae problematis conditioni satisfiat, alia adhuc arbitraria functio,

ex motu initiali cuilibet puncto impresso definienda, accedat, necesse est. En igitur duas rationes, ob quas propositum problema per regulas usitati calculi integralis resolvi non potest: in hoc enim non nisi functiones unius variabilis pertractantur, cum, etsi plures variabiles aequationibus inesse videantur per substitutiones aut transformationes introductae, eae tamen ita a se pendeant, ut omnes per unam possint determinari; deinde quae in calculo integrali communi per integrationes invehuntur novae quantitates, non nisi quantitates constantes sunt, minime vero functiones unius pluriumve variabilium eaeque adeo discontinuae. Huius igitur novae Analyseos vim et proprium characterem Ill. Auctor clarissime et ex primis principiis exposuit, et cum functionum diversae classes commodissimam totius calculi integralis divisionem suppeditent, ex hoc fundamento partes amplissimae huius scientiae distinxit et definivit usumque functionum discontinuarum in Analysi exemplis confirmavit. En igitur campum novum eumque latissime patentem, in quo summa ingenia ad Analyseos incrementa atque feliciores subinde in naturae scrutinio successus vires suas exercere possunt.

1. Quae in Analysi de functionibus, seu quantitatibus per quampiam variabilem utcunque determinatis, tradi solent, ad eas tantum functiones restringuntur, quae continuae vocantur, et quarum formatio certa quadam lege continetur. Ex doctrina linearum curvarum hoc maxime illustratur, ubi applicatae, quatenus per abscissas determinantur, vicem gerunt functionum, ita ut indoles omnium functionum aptissime per lineas curvas repraesentari possit. Ita quomodocunque quantitas y per x determinatur, seu quaecunque functio fuerit y ipsius x, semper curva describi potest, cuius abscissae cuicunque x conveniat ea ipsa applicata y, haecque linea curva congrue naturam illius functionis repraesentare aestimatur. Hinc etiam vicissim proposita linea curva quacunque, eius applicatae certas quasdam functiones abscissarum exhibent, quarum natura in ipsa lineae curvae natura involvitur, dum scilicet cuique abscissae certa respondet applicata, huius valor tanquam functio quaedam abscissae recte spectatur, et quando applicata vel fit imaginaria, vel simul plures valores sortitur, haec ipsa varietas luculentissime ex natura functionis perspicitur.

2. Iam vero notissimum est, in Geometria sublimiori alias lineas curvas considerari non solere, nisi quarum natura certa quadam relatione inter coordinatas per quampiam aequationem expressa definiatur, ita ut omnia eius puncta per eandem aequationem tanquam legem determinentur. Quae lex cum principium continuitatis in se complecti censeatur, quippe qua omnes curvae partes ita vinculo arctissimo inter se cohaerent, ut nulla in illis mutatio

salvo continuitatis nexu locum invenire possit, hanc ob rem istae lineae curvae continuae appellantur, nihilque interest, sive aequatio illarum naturam continens sit algebraica sive transcendens, sive cognita sive etiamnum incognita, dummodo intelligamus dari quandam aequationem, qua natura huiusmodi linearum curvarum exprimatur. Hoc loco non spectatur continuitas tractus, quo rami curvarum porriguntur; ac binae hyperbolae coniugatae aeque lineam curvam continuam constituunt, ac parabola vel ellipsis, etiamsi bini eius tractus penitus a se invicem sint seiuncti. Ob eam enim causam his separatis hyperbolis continuitas tribuitur, quod ambae in una eademque aequatione contineantur, ex eaque formari possint. Atque ex hoc fonte, quae vulgo vage de lege continuitatis disputari solent, interpretari atque ad determinatum significatum revocari conveniret.

3. Constituto continuitatis criterio sponte patet, quid sit functio discontinua, seu lege continuitatis destituta: omnes enim lineae curvae per nullam certam aequationem determinatae, cuiusmodi libero manus tractu delineari solent, tales functiones discontinuas suppeditant, quandoquidem in iis valores applicatarum nulla certa lege abscissis definire licet. Huiusmodi lineae curvae, quatenus superiori generi continuitatis lege definito opponuntur, vulgo mechanicae, aptius vero discontinuae, seu continuitatis lege carentes vocantur, idque non quod earum partes non inter se cohaereant, sed quoniam nulla certa aequatione determinantur. Ita quicunque tractus libera manu super charta ducuntur, etiamsi continuo procedant, tamen secundum hanc definitionem pro discontinuis sunt habendae, siquidem profecto nunquam eveniet, ut huiusmodi tractus certa quadam aequatione contineatur. Atque huc etiam referri convenit lineas vulgo mixtas vocatas, quando partes ex diversis lineis curvis desumtae inter se coniunguntur, vel etiam partes eiusdem lineae alio modo uniuntur. Ita perimeter polygoni ex meris lineis rectis constans aeque huc pertinet, ac lineae ex rectis et arcubus circularibus, vel aliarum quarumcunque curvarum formatae. Etsi enim hic quaevis portio certa quadam aequatione continetur, pro toto tamen tractu nulla aequatio unica, in quo character continuitatis est statuendus, exhiberi potest, quo circa omnes huiusmodi tractus pro lineis discontinuis sunt habendi, perinde ac ii, qui libera manu ducuntur.

4. Iam omnibus huiusmodi lineis et functionibus discontinuis in Analysi geometrica nullum locum concedi, per se est manifestum, cum universa haec speculatio in linearum, quae considerantur, proprietatibus investigandis sit

occupata, quod negotium nullo modo suscipi posset, nisi natura linearum certa quadam lege et aequatione contineretur. Hinc plerique Geometrae hac ratione inducti non dubitaverunt, omnes lineas et functiones discontinuas, tam ex Geometria, quam universa Analysi, penitus proscribere et inter obiecta, a quibus haec scientia abhorreat, detrudere. Hanc sententiam certe palam est professus Celeb. *Alembertus*[1]), cum ego motus cordarum vibrantium ita in genere determinavissem, ut solutio ad omnes motus et figuras, quae cordae initio fuerint impressae, pateret. Mox enim Vir excellentissimus mihi obiecit, motum plane definiri non posse, nisi figura cordae initio impressa fuerit continua ac certa quadam aequatione comprehensa, si secus acciderit et cordae figura initio fuerit discontinua, tum motus secuturi determinationem nullo modo ad Analysin pertinere, atque adeo nefas esse illam investigare velle. Cui obiectioni equidem satis respondi, ac nuper Cel. *La Grange* in Actis Taurinensibus[2]) meam solutionem ita solide propugnavit, ut nulli amplius dubio locus sit relictus.

5. Gravissimi ergo momenti quaestio hic exoritur, quid de functionibus discontinuis, vel lineis sine ulla certa lege descriptis, sit iudicandum, et num et quatenus illis locus in Analysi concedi possit? In problemate certe modo memorato nullum est dubium, quin corda, quae initio ita fuerit diducta, ut eius figura nulla aequatione comprehendi possit, motum sit consecutura, eoque durante singulis momentis ea sit certam figuram et motum receptura, cuius determinatio sane ad Analysin motusque scientiam est referenda, sive fines cognitioni nostrae praescripti huic quaestioni solvendae sufficiant, sive secus. Utroque casu quaestio semper foret omni nostra attentione digna, et cum circa quantitates versetur, ad Analysin certe pertinere est censenda; neque hic quaeritur, quousque sagacitas nostra pateat, cum vix quisquam sit Geometrarum, qui non saepius in quaestionibus vires suas superantibus desudaverit. Neutiquam igitur nefas est putandum, huiusmodi quaestiones attingere; quin potius eo maiori studio in iis esset elaborandum. Omnibus autem difficultatibus diligenter perpensis, etiamnum asseverare audeo, solutionem meam proble-

1) Vide I. Alemberti, Opusc. math. 1, Paris 1761, p. 32, 64; 4, Paris 1768, p. 175. Mém. Acad. Berlin 19, 1763 p. 235/277. Vide quoque notam 2) p. 43 et litteras ab Eulero ad Alembertum 20.6 (20.12) 1763 scriptas (n. 365 indicis Enestroemiani); d'Alembert Opusc. math. 4, p. 146, 162. *Leonhardi Euleri Opera omnia*, series II. H. D.

2) *Lagrange* (1736—1813), *Nouvelles recherches sur la nature et la propagation du son. Addition aux premières recherches sur la nature et la propagation du son.* Misc. Taur. t. II. Oeuvres, 1867, t. I, p. 158 et 319. H. D.

matis de cordis vibrantibus latissimo sensu accepti recte se habere, in eaque felici successu functionum discontinuarum rationem esse habitam. Verum etiam agnosco, hoc problema ad peculiare Analyseos genus adhuc parum excultum esse referendum, cuius generis adeo vis et natura in hoc consistat, ut functiones etiam discontinuas necessario in se complectatur.

6. Ad litem hanc componendam observo, neque in Algebra communi neque in ea Analyseos infinitorum parte, quae adhuc potissimum est tractata, functiones discontinuas admitti posse. Multo latius autem Analysis infinitorum patere atque eiusmodi partes complecti est iudicanda, quae a functionibus discontinuis non solum non abhorreant, sed eas adeo ita natura sua involvant, ut nullum problema eo pertinens rite solutum sit censendum, nisi functiones prorsus arbitrariae, hincque etiam discontinuae, in solutionem fuerint introductae. Istae quidem Analyseos partes etiamnum parum sunt excultae, etiamsi egregia specimina passim reperiantur, neque etiam earum vera indoles satis perspecta videtur. Quare quo hanc indolem luculenter exponam, necesse est, ut varias istas ac diversas Analyseos partes accuratius describam et pro cuiusque indole a se invicem distinguam. Quemadmodum enim vulgo Analysis infinitorum definiri solet, inde vix quicquam lucis ad hoc argumentum illustrandum peti potest, cum pleraeque definitiones maxime sint vagae et confusae, neque subiecti, de quo agitur, naturam satis dilucide ac distincte explicent. Ex quo frequentissimae querelae, quod idea Analyseos infinitorum nusquam accurate descripta ac stabilita reperiatur, fundamento non carent; hic autem illi vitio imprimis est occurrendum, quo diversae huius scientiae partes non satis diligenter a se invicem distinguuntur.

7. Tota autem vis Analyseos infinitorum convenientissime ex notione et indole functionum explicatur, quae commodissime pro numero quantitatum variabilium, per quas certo quodam modo determinantur, in classes distinguuntur. Sic prima classis continebit functiones unicae quantitatis variabilis. Tales functiones sunt applicatae quarumvis linearum, respectu abscissarum. Ita posita abscissa $= x$ et applicata $= y$, erit y functio variabilis x, cuius natura per lineam curvam, seu aequationem, quae inter x et y datur, exprimitur; qua fit, ut statim atque abscissae x determinatus valor tribuitur, etiam applicata y valorem determinatum consequatur, sive is fuerit simplex, sive etiam multiplex, sive etiam imaginarius; unde intelligitur etiam vicissim abscissam x tanquam functionem applicatae y spectari posse. Simili modo si corpus per

quampiam lineam moveatur, eius celeritas in singulis locis etiam ad functiones unicae variabilis est referenda; est quippe functio eius quantitatis variabilis, qua eius lineae puncta continuo determinantur. In hac classe pleraeque quaestiones adhuc tractatae sunt collocandae, etiamsi saepius plures variabiles in computum ingrediantur, siquidem cunctae tandem per unicam determinantur. Veluti si motus lunae investigatur, ad quodvis tempus eius longitudo, latitudo ac distantia a terra quaerenda proponitur; cum autem haec singula elementa tandem per solum tempus determinari debeant, tam longitudo, quam latitudo et distantia, quaeque per se tanquam functio temporis, ideoque unica variabilis spectari poterit.

8. At functiones binarum pluriumve variabilium per eiusmodi binas pluresve variabiles determinantur, quae a se invicem nullo modo pendent, sed cuique seorsim omnes prorsus valores tribuere licet. Tales functiones occurrunt, quando natura solidorum ac superficierum expenditur. Fieri hoc solet per ternas coordinatas x, y et z, quarum binae x et y in plano quodam accipiuntur, tertia vero z huic plano perpendicularis ad superficiem porrigitur. Cum igitur cuique baseos puncto, quod per binas variabiles x et y definitur, certa perpendicularis z immineat, erit utique z functio binarum variabilium x et y a se invicem neutiquam pendentium. Quodsi enim omnia superficiei puncta assignare velimus, tam ipsi x quam ipsi y seorsim omnes prorsus valores tribui oportet, ut hoc modo pro omnibus basis punctis perpendicula illa obtineantur. Deinde si corpus ex particulis heterogeneis sit compositum, ut cuique puncto intra corpus accepto sua peculiaris densitas conveniat, primo quidem situs cuiusque puncti ternis coordinatis x, y et z definitur, nullo modo a se invicem pendentibus, quoniam, ut omnia puncta intra corpus obtineantur, his tribus coordinatis omnes plane valores successive assignari debent. Quare si densitas in quovis puncto quantitate v designetur, ea tanquam functio ternarum variabilium x, y et z spectari debet. Ac si particulae huius corporis motu quocunque agitentur, cuiusque puncti motus non solum ab eius situ ternis coordinatis determinando sed etiam a tempore pendebit, unde motus tanquam functio quatuor variabilium spectari debebit.

9. Constituta hac functionum notione ac divisione, fundamenta Analyseos infinitorum clarissime tradi poterunt, quae disciplina commodissime in tot partes distribuitur, quot dantur functionum classes, propterea quod singulae peculiaribus principiis ac praeceptis sunt superstruendae. Prima igitur pars,

quae fere sola adhuc est exculta et ad quam principia calculi differentialis et integralis potissimum sunt accommodata, circa functiones unicae variabilis versatur. Primo ergo, si y fuerit functio quaecunque unius variabilis x, considerari solent incrementa vel decrementa illius functionis y, dum quantitas x quocunque augmento increscit. Deinde hoc augmentum continuo imminui concipitur, donec tandem prorsus evanescat, quo quidem casu etiam incrementum functionis y in nihilum abit; quae augmenta evanescentia cum differentialia vocentur, evidens est, ea omnia quantitate esse destituta nihiloque adeo aequalia, ita ut de eorum quantitate nulla quaestio institui possit. Neque etiam calculus differentialis in quantitate differentialium, quae nulla est, indaganda occupatur, sed in eorum ratione mutua definienda, quae ratio utique certam obtinet quantitatem. Functionis scilicet y non tam ipsum differentiale dy quam eius ratio ad differentiale dx investigatur, valor nimirum fractionis $\frac{dy}{dx}$, qui quovis casu determinatam quantitatem sortitur et ipse tanquam nova functio ipsius x spectari potest.

10. Cum plerisque haec differentialium notio et rationis, quae inter quantitates evanescentes intercedit, investigatio maxime suspecta videri soleat, unico exemplo omnia dubia evanescent. Proposita igitur sit talis functio $y = axx + bx + c$, ac primum videamus, quantum incrementum haec functio capiat, dum quantitati x augmentum quodcunque ω tribuitur; posito autem $x + \omega$ loco x, functio nostra abit in $axx + 2ax\omega + a\omega\omega + bx + b\omega + c$, ideoque incrementum accipit $= 2ax\omega + a\omega\omega + b\omega$, quod hoc charactere Δy designemus, et ad similitudinem quantitas ω, tanquam augmentum ipsius x, etiam hoc signo Δx denotetur. Cum igitur sit

$$\Delta x = \omega \quad \text{et} \quad \Delta y = 2ax\omega + a\omega\omega + b\omega,$$

erit

$$\frac{\Delta y}{\Delta x} = \frac{2ax\omega + a\omega^2 + b\omega}{\omega} = 2ax + a\omega + b\,,$$

sicque habetur ratio inter incrementa Δx et Δy, quae vera est, quantumvis augmentum ω quantitas x capiat; eadem ergo ratio etiam veritati erit consentanea, si augmentum ω plane evanescens accipiatur, quo casu incrementa illa Δx, Δy his signis dx et dy denotari et differentialia vocari solent; unde perspicuum est, posito $\omega = 0$ prodire

$$\frac{dy}{dx} = 2ax + b,$$

hancque rationem veram esse, etiamsi termini, inter quos subsistit, sint evanescentes. Solum hoc exemplum sufficere videtur omnibus dubiis, quibus vulgo notio infinite parvi in Analysi usurpata impugnari solet, diluendis huicque calculo ab omni suspicione vindicando.

11. Quia haec differentialium ratio $\frac{dy}{dx}$ denuo est functio ipsius x, si ea littera p indicetur, ratio eius differentialis dp ad dx, seu fractio $\frac{dp}{dx}$ simili modo definiri potest, quae, ne opus sit novam litteram in calculum inducere, ob $p = \frac{dy}{dx}$ tali scriptione $\frac{ddy}{dx^2}$ designari solet, quae differentialia secundi gradus involvere dicitur; atque ita porro progrediendo differentialia in has formulas $\frac{d^3y}{dx^3}$, $\frac{d^4y}{dx^4}$ etc. ingredientia tertii, quarti altiorumque ordinum vocantur, quorum significatus, quemadmodum de primo ordine ostendi, semper ad rationem inter differentialia binarum quantitatum, quarum altera alterius est functio, reducitur. Hocque modo omnes controversiae, quae olim circa differentialia omnium ordinum eorumque naturam sunt motae, sponte concidunt, cum quicquid in hoc calculo definitur, semper ad proportionem differentialium, cuius realitas nulli dubio est subiecta, revocetur, neque amplius veritates per hunc calculum erutae Geometricis ullo pacto postponendae videbuntur. Equidem non diffiteor, eiusmodi rationes loquendi in hac disciplina esse receptas, quae differentialibus quantitatem quampiam valde exiguam tribuere videantur, sed cum earum significatio semper ex stabilitis principiis sit interpretanda, tales loquendi formulas, etsi minus congruas, tolerari convenit. Quin etiam cum expressio $p = \frac{dy}{dx}$ prorsus sit realis, etiam haec aequalitas $dy = pdx$ merito admittitur, tametsi in neutro membro ulla quantitas agnoscitur.

12. Haec igitur definitio calculi differentialis nullis amplius tenebris est involuta, qua is vocatur methodus, proposita quacunque functione unius pluriumve variabilium, rationes, quae inter differentialia tam primi quam altiorum ordinum intercedunt, investigandi. De functionibus quidem unius variabilis, ad quas solas hic etiamnum respicio, ista definitio maxime est perspicua; si enim y fuerit functio quaecunque ipsius x, calculus differentialis

docet, quomodo valor fractionis $\frac{dy}{dx}$ sit eliciendus eademque regula, qua hoc praestatur, valet quoque pro differentialibus altioribus, cum, posito $\frac{dy}{dx} = p$, ex hac etiam functione ipsius x eadem methodo valor $\frac{dp}{dx}$ seu $\frac{ddy}{dx^2}$ obtineatur, ac si ulterius statuatur $\frac{dp}{dx} = \frac{ddy}{dx^2} = q$, item $\frac{dq}{dx} = \frac{d^3y}{dx^3} = r$, tum $\frac{dr}{dx} = \frac{d^4y}{dx^4} = s$ etc., eadem methodus sufficit his omnibus valoribus q, r, s etc. inveniendis; atque huc sunt trahenda, quae vulgo de calculo differentio-differentiali et differentialibus altiorum ordinum tradi solent, quae, si rite intelligantur, nihil sane continent, quod primis nostrae cognitionis principiis adversetur. Quando etiam in elementis calculi differentialis saepe plures quantitates variabiles occurrunt et praecepta ad eas partes, quas constituo sequentes, referenda videantur, tamen semper in eas certus quidam nexus admittitur, ut tandem omnes tanquam functiones unius variabilis spectari queant. Interim tamen regulae differentiandi sequentium partium a prima non discrepant.

13. Calculum autem integralem in genere ita definio, ut sit methodus inveniendi indolem functionum ex data quacunque differentialium relatione; quam definitionem pro casu functionum unicae variabilis ante clarius evolvam, quam ad functiones plurium variabilium sum progressurus. Posito scilicet pro functione unius variabilis $\frac{dy}{dx} = p$, $\frac{dp}{dx} = q$, $\frac{dq}{dx} = r$ etc., si aequatio proponatur quaecunque, in quam, praeter quantitates x et y, etiam istae p, q, r etc. ex differentialibus ortae ingrediantur, in hoc officium calculi integralis versatur, ut ex ista aequatione seu relatione differentialium data natura functionis y, quemadmodum scilicet per x determinatur, eliciatur; quae operatio vocari solet integratio. Plurimum autem abest, quo minus haec methodus adhuc satis sit elaborata, et si omnes quaestiones in eam cadentes perpendamus, paucissimas eius ope resolvere licet; variis autem ea, quatenus est exculta, continetur praeceptis pro ordine differentialium, quae in relationem datam ingrediuntur. Ita si proponatur relatio quaecunque inter quantitates x, y et $p = \frac{dy}{dx}$, quae aequatio primi gradus differentialis vocatur, pluribus quidem casibus integratio succedit; sin autem ea relatio insuper quantitatem q involvat, aequatio vocatur differentialis secundi gradus, duplicique integratione opus est, antequam ad desideratam relationem inter x et y, unde ratio istius functionis y innotescat, perveniatur. Hic multo pauciores sunt casus, quibus ad scopum

pertingere licet; simulque patet, quid de aequationibus differentialibus tertii altiorumque ordinum sit iudicandum.

14. Verum circa has integrationes, quae functionibus unius tantum variabilis investigandis inserviunt, singularis quaedam affectio, qua istius methodi praecipua indoles continetur, probe est observanda. Affectio autem ista in hoc consistit, quod aequatio integrata semper novam quandam constantem quantitatem recipiat, cuius in aequatione differentiali ne vestigium apparet, hancque quantitatem constantem prorsus arbitrio nostro relinqui. Ita si habeatur ista aequatio differentialis

$$\frac{dy}{dx} = 2ax + b \quad \text{seu} \quad dy = 2axdx + b\,dx,$$

ubi quidem litterae a et b denotant quantitates constantes datas, aequatio integralis in omni extensione ita se habet:

$$y = axx + bx + C,$$

ubi C designat quantitatem constantem a praecedentibus minime pendentem, et cuius valor penitus arbitrio nostro relinquitur, neque integratio cuiusquam aequationis differentialis pro completa et perfecta est habenda, nisi huiusmodi quantitas constans arbitraria fuerit introducta. Simili modo si relatio proposita involvat differentialia secundi gradus, quoniam duplici opus est integratione, solutio completa duas eiusmodi constantes arbitrarias complecti debet; tres vero eiusmodi constantes requiruntur, si aequationes differentiales tertii gradus perfecte resolvuntur. De his autem constantibus id praecipue est notandum, quod cum natura problematum arctissimo nexu cohaereant atque omnia problemata, quorum resolutio ad aequationes differentiales perducitur, ita sint comparata, ut post peractam integrationem constantes illae ingressae ex ipsa rei natura et circumstantiis adiunctis determinationem suam adipiscantur.

15. His igitur constat prima pars Analyseos infinitorum, quae circa functiones unius tantum variabilis versatur, atque ex his multo facilius intelligetur, quid de reliquis partibus, in quibus functiones duarum pluriumve variabilium, sit tenendum. In differentialibus autem iam diversa deprehenditur ratio, cum ea hic non absolute inter se comparare liceat. Si enim z fuerit functio quaecunque binarum variabilium x et y, circa differentiationem quaestio bipartita

est statuenda: primo scilicet quaeritur differentiale ipsius z, dum, servante y eundem valorem, altera variabilis x differentiali suo dx augetur, ut inde valor fractionis $\frac{dz}{dx}$ obtineatur; simili modo tractata x ut constante, altera y incrementum dy capere assumitur, et collecto inde incremento dz, quo functio z augetur, fractio $\frac{dz}{dy}$ rationem differentialem ex variabilitate solius quantitatis y natam exprimit. Utraque autem haec fractio $\frac{dz}{dx}$ et $\frac{dz}{dy}$, uti casu praecedente, meris terminis finitis continebitur, et ambae tanquam novae functiones binarum variabilium x et y spectari poterunt. Inventis autem his binis valoribus vera ratio differentialis functionis propositae z perspicitur; ex iis enim coniunctis demum patet, quomodo differentiale ipsius z ratione variabilitatis utriusque quantitatis x et y se habeat. Hanc distinctionem ipsa rei natura postulat, sine qua ratio differentiationis huiusmodi functionum ne intelligi quidem posset, quae autem nunc per se est manifesta.

16. Quicquid igitur ad differentiationem functionum duarum variabilium spectat, id totum ad binas istas formulas $\frac{dz}{dx}$ et $\frac{dz}{dy}$ reducitur, quarum valores quovis casu terminis finitis per ambas variabiles x et y exprimuntur. Ne autem huiusmodi fractiones cum praecedentibus confundantur, uncinulis includi hocque modo

$$\left(\frac{dz}{dx}\right) \text{ et } \left(\frac{dz}{dy}\right)$$

scribi solent. Quodsi has fractiones litteris p et q designemus, erit utique

$$dz = p\,dx + q\,dy$$

differentiale completum functionis z, et quoniam p et q iterum ut functiones ipsarum x et y spectari possunt, intelligitur etiam, quid sibi velint hae formulae:

$$\left(\frac{dp}{dx}\right)=\left(\frac{ddz}{dx^2}\right), \quad \left(\frac{dp}{dy}\right)=\left(\frac{ddz}{dx\,dy}\right), \quad \left(\frac{dq}{dx}\right)=\left(\frac{ddz}{dx\,dy}\right) \text{ et } \left(\frac{dq}{dy}\right)=\left(\frac{ddz}{dy^2}\right),$$

quae differentialia secundi gradus in se complectuntur, similique modo progressio fit ad differentialia altiorum graduum. Proposita ergo functione quacunque z binarum variabilium x et y, calculus differentialis regulas praescribit,

quibus valores omnium istarum formularum differentialium inveniri queant; primo scilicet primi gradus, quae sunt

$$\left(\frac{dz}{dx}\right) \text{ et } \left(\frac{dz}{dy}\right),$$

tum secundi gradus, quae sunt

$$\left(\frac{ddz}{dx^2}\right), \left(\frac{ddz}{dx\,dy}\right) \text{ et } \left(\frac{ddz}{dy^2}\right),$$

porro tertii gradus, quae sunt

$$\left(\frac{d^3z}{dx^3}\right), \left(\frac{d^3z}{dx^2dy}\right), \left(\frac{d^3z}{dx\,dy^2}\right), \left(\frac{d^3z}{dy^3}\right)$$

et sic deinceps. Ubi quidem notandum, ipsam methodum has formulas definiendi a prima parte non discrepare, cum in qualibet differentiatione unica tantum quantitas pro variabili habeatur. Superfluum foret, haec eadem momenta de functionibus trium pluriumve variabilium exponere, quippe quae ex allatis iam satis sunt perspicua.

17. Integralis autem calculi munus in hoc consistit, ut proposita relatione quacunque inter quantitates x, y, z et formulas differentiales modo allatas inde natura functionis z, quemadmodum ex variabilibus x et y conflatur, investigetur. Relatio vero illa data per aequationem exprimitur, quae, si tantum formulas differentiales primi ordinis

$$\left(\frac{dz}{dx}\right) \text{ et } \left(\frac{dz}{dy}\right)$$

praeter quantitates ipsas x, y et z complectitur, aequatio differentialis primi gradus vocatur; sin autem in eam insuper formulae differentiales secundi ordinis

$$\left(\frac{ddz}{dx^2}\right), \left(\frac{ddz}{dx\,dy}\right), \left(\frac{ddz}{dy^2}\right)$$

vel porro tertii altiorumve ordinum ingrediantur, tum aequatio illa eiusdem ordinis differentialis dicitur. Haecque est forma generalis calculi integralis, quatenus circa functiones duarum variabilium est occupatus; ex quo simul

intelligitur, quomodo reliquae partes Analyseos infinitorum, in quibus functiones trium pluriumve variabilium tractantur, sint definiendae. At calculus integralis ad functiones duarum variabilium accommodatus plurimum differt a calculo integrali communi, ubi nonnisi functiones unius variabilis occurrunt, et praecepta omnino singularia postulat, praeterquam quod in eo omnia quoque artificia prioris partis sint in usum vocanda. Verum haud diu est, ex quo haec pars Analyseos coli est coepta, ita ut vix adhuc prima eius elementa satis sint evoluta. Eximia quidem huius calculi specimina iam passim reperiuntur, quorum tractatio autem minus ad praecepta calculi communis est adstricta; unde latissimus campus aperitur, in quo summa ingenia ad maximum scientiae incrementum vires suas exercere poterunt.

18. Huius autem novi calculi vis et quasi proprius character minime adhuc satis perspectus videtur. Quemadmodum enim calculi integralis communis vis in eo consistit, ut qualibet integratione nova quantitas constans arbitrio nostro permissa in calculum introducatur; ita in hac parte, circa functiones binarum variabilium occupata, singulis integrationibus non solum nova quantitas constans, sed adeo nova functio cuiuspiam variabilis prorsus indeterminata in calculum invehitur, quae ita ab arbitrio nostro pendet, ut eius loco etiam functiones discontinuae assumi queant. Quare functionum discontinuarum usus ab hoc fere novo calculi genere nonsolum non excluditur, sed etiam quasi essentialiter ad eius naturam pertinere sit iudicandus; neque etiam ulla integratio in hoc calculo pro completa et absoluta est habenda, nisi in aequationem integralem huiusmodi functio prorsus arbitraria fuerit introducta; ac si aequatio differentialis proposita fuerit secundi altiorisve gradus, ita ut binis pluribusve integrationibus sit opus, necesse est, totidem functiones arbitrariae in ultima aequatione integrali reperiantur, quod nisi eveniat, integrale non magis pro completo haberi potest, quam in calculo integrali ordinario, ubi introductio constantium arbitrariarum negligitur. Quando autem de functionibus trium variabilium agitur, qualibet integratione functio arbitraria binarum variabilium in calculum introducitur; qua circumstantia iste calculus a praecedentibus ita distinguitur, ut genus peculiare constituere sit censendus, cum natura cuiusque generis, ex indole quantitatis arbitrariae per integrationem invectae, convenientissime diiudicetur. Tum vero si quaestio circa functiones quatuor variabilium versatur, haec quantitas arbitraria quavis integratione introducenda fit functio trium variabilium et ita porro.

19. Haec autem neutiquam calculi cuipiam morositati sunt tribuenda, quae omni usu destituantur et inani speculationi tantummodo inserviant, sed potius naturae rerum maxime innituntur et cum veritatum concatenatione pulcherrime cohaerent. Quemadmodum enim omnia problemata circa functiones unicae variabilis, cuiusmodi sunt fere omnia, quae adhuc in Analysi sunt tractata, perfecte non solvuntur, nisi qualibet integratione nova quantitas constans inferatur, quam deinceps ex circumstantiis problematis determinari oportet; ita etiam omnia problemata, quorum solutio ad functiones binarum variabilium perducitur, natura sua ita sunt comparata, ut, nisi quavis integratione nova functio arbitraria seu indefinita unius variabilis induceretur, omnibus conditionibus problema determinantibus nullo modo satisfieri posset. Eximium huius rei specimen cernitur in problemate de cordis vibrantibus; si enim cuiusque cordae puncti, quod ab altero termino distat intervallo $= x$, pro tempore elapso $= t$, elongatio ab axe seu statu aequilibrii ponatur $= z$, evidens est, z esse functionem duarum variabilium t et x, quoniam utique ista elongatio, tam pro diversis cordae punctis, quam pro temporis fluxu, variatur. Cum igitur posito tempore $t = 0$ is cordae status prodire debeat, qui ipsi initio fuerit inductus, et ubi elongatio z functioni cuidam datae intervalli x erat aequalis, solutio perfecta esse nequit, nisi huiusmodi functionem indefinitam complectatur, quae deinceps ex statu cordae initiali definiri queat; et quoniam iste status ab arbitrio nostro ita pendet, ut cordae figura quaecunque irregularis et discontinua induci potuerit, etiam functio illa per Analysin introducta ita late patere debet, ut etiam discontinua, seu a continuitatis lege abhorrentia, in se complectatur.

20. Ne autem hic ulli dubio locus relinquatur, eiusmodi problema evolvam, cuius solutio adeo ex elementis facile deducitur, et quae ita est comparata, ut in ea functiones discontinuae, seu lineae curvae pro lubitu ductae, necessario admitti debeant; deinde idem problema analytice expediam, quo clarius necessitas functionum arbitrariarum, quae integratione introducuntur, ex consensu cum priori solutione, elucescat. Problema autem ita se habet: *ut omnia solida, ad quorum superficiem in singulis punctis normales ductae eiusdem sint quantitatis,* [*inveniantur*]. Quando de lineis agitur, notum est, praeter circulum nullam dari lineam curvam, cuius omnes normales sint inter se aequales; at si haec aequalitas normalium ad solida extenditur, ut omnes rectae a dato plano tanquam basi ad superficiem normaliter ductae debeant esse inter se aequales, infinita exhiberi possunt solida, in quae haec proprietas

competit. Primo scilicet hoc manifesto evenit in hemisphaerio, vel etiam sphaera, cuius centrum in plano illo seu basi est situm, dum omnes rectae normales simul sunt radii sphaerae. Deinde si cylindrus ita collocetur, ut eius axis in basem incidat, omnes quoque normales inter se aequales habentur. Hinc autem colligitur solutio multo latius patens, quoniam salva hac proprietate axis cylindri quomodocunque incurvari potest, quam solutionem generalem ita enunciare licet. Descripta super plano fixo linea quacunque curva, sive continua sive discontinua, eiusmodi solidum super ea exstruatur, cuius omnes sectiones, ad illam lineam normaliter factae, sint semicirculi, quorum centra in eam lineam incidant. Nisi ergo solutio analytica pariter ita late pateat, ut lineam pro lubitu ductam, seu, quod eodem redit, functionem indefinitam in se contineret, ea certe pro perfecta et absoluta haberi non posset.

21. Positis igitur binis coordinatis in plano fixo assumtis x et y, perpendiculo autem inde ad superficiem quaesitam pertingente $= z$, quia z consideratur ut functio binarum variabilium x et y, statuantur formulae differentiales

$$\left(\frac{dz}{dx}\right) = p \quad \text{et} \quad \left(\frac{dz}{dy}\right) = q\,,$$

ut sit $dz = pdx + qdy$. Hinc autem normalis in superficiem ad planum usque fixum porrecta colligitur

$$= z\sqrt{(1 + pp + qq)}\ ;$$

quae quia debet esse constantis magnitudinis, ponatur

$$z\sqrt{(1 + pp + qq)} = a$$

et pro z eiusmodi investigari oportet functionem binarum variabilium x et y, ut haec conditio, quae est aequatio differentialis primi gradus, impleatur. Quo facilius autem ad resolutionem per integrationem perveniamus, his utamur substitutionibus: sit

$$p = \frac{\sin.\Phi \cos.\omega}{\cos.\Phi} \quad \text{et} \quad q = \frac{\sin.\Phi \sin.\omega}{\cos.\Phi}\,,$$

ut fiat

$$pp + qq = \frac{\sin.\Phi^2}{\cos.\Phi^2}\,,$$

hincque

$$\frac{z}{\cos.\Phi} = a \quad \text{seu} \quad z = a\cos.\Phi,$$

unde aequatio differentialis assumta transformatur in hanc:

$$-\,ad\Phi\sin.\Phi = \frac{\sin.\Phi}{\cos.\Phi}(dx\cos.\omega + dy\sin.\omega)$$

seu

$$-\,ad\Phi\cos.\Phi = dx\cos.\omega + dy\sin.\omega,$$

ubi cum pars prior integrationem admittat, etiam pars posterior integrabilis est reddenda, qua conditione certa relatio inter variabiles x, y et ω stabilitur. Cum igitur integrando obtineamus:

$$-\,a\sin.\Phi = x\cos.\omega + y\sin.\omega - \int d\omega\,(y\cos.\omega - x\sin.\omega),$$

evidens est, hoc integrale exhiberi non posse, nisi formula $y\cos.\,\omega - x\sin.\,\omega$ fuerit functio unius variabilis ω[1]). Statuatur ergo

$$y\cos.\omega - x\sin.\omega = F'\colon\omega,$$

ut fiat

$$\int d\omega\,(y\cos.\omega - x\sin.\omega) = F\colon\omega,$$

eritque

$$-\,a\sin.\Phi = x\cos.\omega + y\sin.\omega - F\colon\omega.$$

Vel denotet Ω functionem quamcunque ipsius ω utcunque indefinitam, ut etiam functiones discontinuae inde non excludantur, et posito $F'\colon\omega = \Omega$, erit $F\colon\omega = \int\Omega\,d\omega$, et problematis solutio ob $a\sin.\Phi = \sqrt{(aa - zz)}$ his aequationibus continetur:

$$y\cos.\omega - x\sin.\omega = \Omega \quad \text{et} \quad \sqrt{(aa - zz)} = \int\Omega\,d\omega - x\cos.\omega - y\sin.\omega,$$

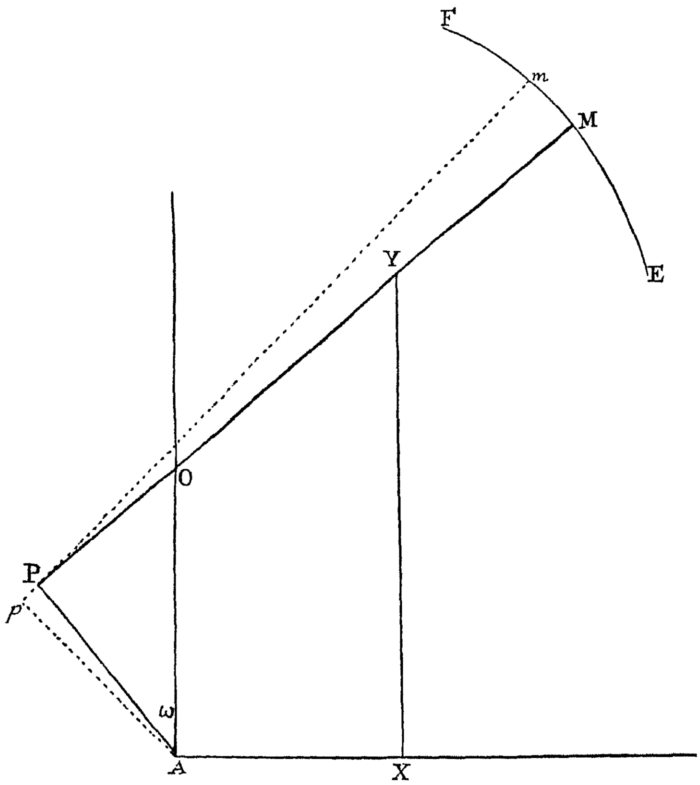
ubi quidem signum radicale aeque negative ac positive capi potest.

22. Videamus iam, quomodo hae formulae ad constructionem perduci queant. Referat ipsa tabula planum illud, fixam basin corporis quaesiti con-

1) Vide *L. Euleri* Commentationem 285 (indicis *Enestroemiani*) § 10. *Investigatio functionum ex data differentialium conditione.* Novi comment. acad. sc. Petrop. 9, 1764, p. 175. Vide quoque *Institutionum calculi integralis* vol. III, § 77. *Leonhardi Euleri Opera omnia,* series I, vol. 22, 13. H.D.

stituens, in quo sint (vide fig.) binae coordinatae $AX = x$ et $XY = y$, ita ut puncto Y perpendiculariter immineat tertia coordinata z. In ipso isto plano axi AX normaliter iungatur recta AO ducaturque recta AP, ita ut sit angulus $OAP = \omega$, ad hanc ex Y agatur normalis YP, eritque

$$AP = y \cos.\omega - x \sin.\omega \quad \text{et} \quad PY = y \sin.\omega + x \cos.\omega.$$



Quibus lineis in calculum introductis ambae nostrae aequationes ita se habebunt:

$$AP = \Omega \quad \text{et} \quad PY + \sqrt{(aa - zz)} = \int \Omega \, d\omega.$$

ucatur ergo recta PY in M, ut sit $YM = \sqrt{(aa - zz)}$, fiatque propterea $= \int \Omega \, d\omega$, quae relatio inter lineas AP et PM probe est perpendenda. Cum iam in situ proximo, angulo scilicet OAP aucto suo differentiali $PAp = d\omega$, sit arculus radio AP descriptus $Pp = \Omega d\omega$, hic arculus simul differentiale lineae MP exhibebit, ita ut sit $pm = PM + Pp$, ex quo intelligitur, rectam PM altero termino M in eiusmodi curva EMF terminari, in quam ea iugiter

sit normalis, qua proprietate tota nostra solutio analytica continetur. Quocirca constructio quaesita ita erit comparata: Descripta pro lubitu curva quacunque EMF, quae sive sit continua sive secus nihil refert, ad singula eius puncta M ducantur normales MP, utrinque producendae, et ex harum rectarum singulis punctis Y verticaliter erigantur perpendicula $YZ = z$, ut sit $YZ^2 + MY^2 = aa$, quod praestabitur, si singulis centris M radio $= a$, cui normales debent esse aequales, describantur circuli in planis ad basin ipsamque curvam EMF normalibus; horum enim circulorum peripheriae in ipsa superficie corporis quaesiti erunt positae, et rectae in hanc superficiem normales omnes erunt $= a$, et in lineam EMF pro lubitu ductam incident.

23. Levissima autem attentione adhibita perspicuum est, hanc constructionem ex solutione analytica petitam prorsus congruere cum superiori constructione, quam sola elementorum consideratio suppeditaverat. Manifestum enim est, corpus ita fore comparatum, ut omnes eius sectiones ad lineam EMF normaliter factae sint circuli inter se aequales, centra sua in ipsa hac linea habentes. Ob utriusque autem solutionis consensum hoc imprimis notandum est, solutionem analyticam non futuram esse completam, nisi functio Ω per integrationem ingesta latissime pateret, atque adeo omnes omnino valores, tam continuos, quam discontinuos, in se complecteretur, quandoquidem linea illa EMF, functioni Ω respondens, penitus arbitrio nostro relinquitur, ut etiam lineas libero manus tractu ductas in usum vocare liceat. Quod autem de hoc problemate est ostensum, simul de omnibus aliis eiusdem generis valet, quorum scilicet solutio functiones binarum variabilium implicat, ex quo quaestio initio proposita de usu functionum discontinuarum in Analysi ita est resoluta, ut in Analysi quidem communi, quae circa functiones unius variabilis tantum versatur, huiusmodi functionibus nullus locus sit concedendus, in sublimioribus autem Analyseos partibus, ubi functiones binarum pluriumve variabilium tractantur, tales functiones ita necessario ad calculi essentiam pertinere sint censendae, ut nulla integratio pro absoluta et completa haberi queat, nisi simul functio maxime indefinita, atque adeo etiam discontinua, in calculum introducatur.

DE VARIIS INTEGRABILITATIS GENERIBUS

Commentatio 429 indicis ENESTROEMIANI

Novi Commentarii academiae scientiarum Petropolitanae 17 (1772), 1773, p. 70—104

Summarium ibidem p. 9—11

SUMMARIUM

Multiplicatorum, per quos formulae differentiales reddantur integrabiles, investigatio tot saepenumero tantisque involvitur difficultatibus, ut hoc ipsum argumentum eo magis summorum Geometrarum studio dignum censeri debeat, quo praeclariorem eiusmodi multiplicatores usum praestant in resolvendis aequationibus altiorum graduum, ad quarum integrationem, ab his subsidiis si discesseris, vix ullus aditus patere videbatur. Quum autem statim, ac unus quispiam multiplicator fuerit cognitus, quem utpote in quovis genere simplicissimum recte primitivum appellare licet, ex eo infinite multos alios derivare liceat, quaestio sine dubio maximi momenti inde nascitur de invenienda expressione generali, quae omnes plane multiplicatores possibiles eiusdem formulae in se complectatur; atque in hoc consistit praecipuum momentum, quod Ill. EULERUS in praesenti dissertatione meditationibus suis est prosecutus, dum pluribus selectis exemplis ostendit, semper duos multiplicatores primitivos locum habere, et nonnunquam usu quoque venire, ut plures multiplicatores videantur diversi, qui tamen ad duos queant revocari; et cum huiusmodi multiplicatorum inventio quandoque etiam Analyseos vires prorsus superet, eo maiori attentione digna est ea methodus, quam ab Ill. Viro in hoc scripto ad hoc institutum legimus adplicatam, cuius ope plurimis casibus tales multiplicatores invenire licet, quaeque insignem usum habet in resolvendis aequationibus secundi gradus, quas quippe omnes ad hanc formam redigere licet:

$$Pdp + Qdx + Rdy = 0,$$

posito $dy = pdx$, ubi manifestum est, si unus huius formulae multiplicator innotuerit, statim obtineri aequationem semel integratam adeoque primi ordinis; at vero si bini multiplicatores fuerint cogniti, tum statim aequationem bis integratam seu finitam elici posse, ita ut integratione repetita plane non fuerit opus.

1. Si quantitas variabilis p absolute spectetur et quaeratur, quomodo quantitatem V comparatam esse oporteat, ut formula Vdp fiat integrabilis, tunc nullum est dubium, quin ista quantitas V debeat esse functio quaepiam ipsius p. Vocabulum enim integrabilitatis ita hic sensu latissimo accipio, ut, quaecunque functio ipsius p fuerit V, formulam Vdp semper integrabilem esse dicam, nihilque intersit, sive eius integrale algebraice, sive per logarithmos, sive per arcus circulares, sive per quascunque altiores quantitates transcendentes exprimatur, quandoquidem formulam integralem $\int Vdp$ semper per quadraturam cuiuspiam curvae exhibere licet.

2. Longe aliter autem se res habet, quando quantitas p certa quadam ratione ad alias quantitates variabiles refertur; tum enim praeter functiones ipsius p quantitati illi V etiam alios valores tribuere licet, quibus formula Vdp integrabilis redditur. Veluti si p ita ad binas coordinatas x et y referatur, ut sit

$$dy = pdx \quad \text{sive} \quad p = \frac{dy}{dx},$$

tum loco V etiam sumi poterit x, quoniam formula xdp revera est integrabilis, quum enim sit

$$\int xdp = px - \int pdx, \quad \text{ob} \quad \int pdx = y$$

erit utique

$$\int xdp = px - y.$$

3. Quin etiam idem locum habet in ipsis differentialibus primitivis dx et dy, ut enim formula Vdx sit integrabilis, hoc non solum usu venit, si fuerit V functio quaecunque ipsius x, sed etiam casu, quo $V = p$, quum sit

$$\int pdx = y,$$

simili modo, ut formula $\int Vdy$ fiat integrabilis, loco V non solum functio quaecunque ipsius y accipi poterit, sed etiam casu

$$V = \frac{1}{p} \quad \text{fit} \quad \int \frac{dy}{p} = x.$$

4. Quo haec generalius prosequamur, vocemus quantitatem V multiplicatorem, per quem quaepiam formula differentialis reddatur integrabilis, unde ex praemissis patet, si formula differentialis fuerit

$$\text{vel } dp, \text{ vel } dx, \text{ vel } dy,$$

tum multiplicatorem esse

$$\text{vel}\quad V = x\,,\quad \text{vel}\quad V = p\,,\quad \text{vel}\quad V = \frac{1}{p}\,.$$

5. Quantumvis haec facilia et obvia videantur, tamen saepenumero investigatio huiusmodi multiplicatorum maxime ardua deprehenditur, et quod eius usum maxime commendat, saepius hoc modo aequationes differentiales, tam secundi, quam altiorum graduum, satis commode resolvere licet, ad quas absque his subsidiis vix alius aditus patere videatur. Saepius enim iam notavi, praecipuum negotium in aequationibus differentialibus integrandis ad inventionem idoneorum multiplicatorum reduci, ex quo investigatio huiusmodi multiplicatorum sine dubio maximi momenti censeri debet[1]).

6. Haec accuratius perscrutandi occasionem mihi dedit haec quaestio, qua linea curva inter binas coordinatas x et y contenta quaeritur, cuius radius osculi aequalis sit futurus lineae rectae

$$\sqrt{(xx + yy)}\,,$$

quum enim posito $dy = p\,dx$ sit radius osculi

$$= \frac{dx}{dp}(1 + pp)^{\frac{3}{2}},$$

haec habetur aequatio resolvenda

$$\frac{dp}{(1 + pp)^{\frac{3}{2}}} = \frac{dx}{\sqrt{(xx + yy)}}\,,$$

cuius utrumque membrum non sine admiratione integrabile fieri deprehendi ope multiplicatoris $x + py$; tum enim pro posteriori membro fit

$$\frac{dx\,(x + py)}{\sqrt{(xx + yy)}} = \frac{x\,dx + y\,dy}{\sqrt{(xx + yy)}}\,,$$

1) Cf. L. EULERI Commentationem 265 indicis *Enestroemiani. De aequationibus differentialibus secundi gradus.* Novi Comment. acad. sc. Petrop. 7, 1761, p. 163. Cf. quoque *Institutiones calculi integralis,* vol. II, § 865—928. *LEONHARDI EULERI Opera omnia,* series I, vol. 22, p. 295 et 12, p. 94. Vide porro Commentationes 431, 700 huius voluminis. H. D.

cuius integrale est

$$\sqrt{(xx+yy)};$$

pro priori vero membro res non adeo est manifesta; posito autem

$$y = px + v,$$

ut fiat

$$dy = pdx = pdx + xdp + dv, \quad \text{ideoque} \quad dv = -xdp,$$

habebimus

$$dp(x+py) = xdp + ypdp = x(1+pp)dp + vpdp = -dv(1+pp) + vpdp,$$

ita ut prius membrum fiat

$$\frac{-dv(1+pp)+vpdp}{(1+pp)^{\frac{3}{2}}},$$

cuius integrale manifesto est

$$= -\frac{v}{\sqrt{(1+pp)}} = \frac{px-y}{\sqrt{(1+pp)}},$$

sicque tota aequatio integralis erit

$$\sqrt{(xx+yy)} + C = \frac{px-y}{\sqrt{(1+pp)}}.$$

7. Haec igitur perpendens non amplius dubitavi, quin omnes huiusmodi formulae differentiales duplicem admittant multiplicatorem, ita ut, si talis formula iam per se sit integrabilis, quae in genere sit dv, praeter multiplicatores naturales, qui sunt functiones ipsius v, etiam dentur multiplicatores alius indolis, qui non sint functiones ipsius v, quemadmodum in exemplis allatis fieri vidimus.

8. Statim autem ac unus quispiam multiplicator fuerit cognitus, ex eo mox infinitos alios multiplicatores concludere licet; ita quum formulae dx multiplicator sit p et $\int pdx = y$, tum functio quaecunque ipsius y, quae sit Y, per p multiplicata dabit etiam multiplicatorem idoneum, nam dx multiplicatum in Yp dat Ydy, quod manifesto est integrabile. Deinde si X denotet functionem quamcunque ipsius x, formula dy multiplicatorem habebit $\frac{X}{p}$, tum enim prodit

$$\frac{Xdy}{p} = Xdx.$$

Simili modo quum sit

$$\int x dp = px - y,$$

si V denotet functionem quamcunque formulae $px - y$, tum Vx erit multiplicator ipsius dp, si quidem erit

$$Vxdp = V \cdot d(px - y);$$

huiusmodi autem multiplicatores, qui hoc modo ex uno quodam multiplicatore cognito concluduntur, omnes eiusdem generis sunt censendi, unde simplicissimum eorum in quovis genere primitivum appellabo, quippe quo cognito, reliqui omnes innotescunt.

9. Ita si in genere proposita sit huiusmodi formula differentialis:

$$Pdp + Qdx + Rdy,$$

ubi P, Q, R sint functiones quaecunque ipsarum x, y et p, quae integrabilis reddatur ope multiplicatoris M, sequenti modo omnes reliqui multiplicatores eiusdem generis inveniri poterunt. Ponatur

$$M(Pdp + Qdx + Rdy) = dv,$$

ita ut dv sit verum differentiale, ac denotet V functionem quamcunque ipsius v, manifestumque erit, multiplicatorem quoque fore VM, si quidem tum habebitur

$$VM(Pdp + Qdx + Rdy) = Vdv,$$

quae formula per hypothesin est integrabilis.

10. Simili modo, si pro eadem formula proposita

$$Pdp + Qdx + Rdy$$

adhuc alius multiplicator primitivus N fuerit repertus, tum ex eo etiam infiniti alii eiusdem generis erui poterunt, ita ut hoc modo duae inveniantur formulae generales pro multiplicatoribus formulae differentialis propositae. Hinc ergo ista quaestio maximi momenti nascitur, quam seorsim proponi operae pretium erit.

PROBLEMA

Si formula

$$Pdp + Qdx + Rdy$$

integrabilis fiat tam per multiplicatorem M, quam per alium diversae naturae N, invenire expressionem generalem, quae omnes plane multiplicatores possibiles eiusdem formulae in se complectatur.

SOLUTIO

Quum M et N sint multiplicatores, ponamus

$$M(Pdp + Qdx + Rdy) = dv$$

et

$$N(Pdp + Qdx + Rdy) = du$$

eruntque quantitates v et u cognitae functiones, iam denotet z functionem quamcunque binarum harum variabilium v et u, cuius differentiale propterea huiusmodi formam habebit:

$$dz = Sdv + Tdu,$$

ubi functiones S et T ex functione z erunt cognitae; hac forma iam inventa, dico expressionem generalem, omnes plane multiplicatores in se complectentem, fore

$$= SM + TN;$$

tum enim habebitur

$$(SM + TN)\ (Pdp + Qdx + Rdy) = Sdv + Tdu = dz,$$

cuius integrale per hypothesin est z, ubi pro z functio quaecunque binarum variabilium v et u pro lubitu sumi potest.

11. Ut hoc exemplo illustremus, sit proposita formula dp, cuius duo multiplicatores constant

$$\begin{array}{lll} M = 1 & \text{et} & N = x, \text{ hinc ergo fit} \\ dp = dv & \text{et} & xdp = du, \text{ ideoque} \\ v = p & \text{et} & u = px - y, \end{array}$$

quare si z denotet functionem quamcunque harum duarum variabilium v et u, sitque

$$dz = Sdv + Tdu,$$

multiplicator universalis erit $S + Tx$.

12. Circa hanc formulam observandum est, non absolute necessarium esse, ut valores litterarum S et T ex certa quadam functione z deriventur. Dummodo enim pro litteris S et T eiusmodi functiones ipsarum v et u capiantur, ut sit $\left(\frac{dS}{du}\right) = \left(\frac{dT}{dv}\right)$, tum enim semper formula $SM + TN$ erit multiplicator idoneus formulae differentialis

$$Pdp + Qdx + Rdy,$$

et producti integrale erit ipsa functio illa z, quam ex litteris S et T facile invenire licet.

13. Quod autem ista formula $SM + TN$ omnes plane multiplicatores formulae differentialis propositae in se complectatur, ratio in eo est sita, quod semper duo tantum eiusmodi multiplicatores primitivi M et N exhiberi queant, qui a se invicem non pendeant; si enim plures eiusmodi multiplicatores locum haberent, tum forma ista utique non foret generalis, sed alia multo generalior exhiberi posset; ratio autem, cur duo tantum huiusmodi multiplicatores locum inveniant, in eo est quaerenda, quod inter ternas nostras variabiles, x, y et p unica detur relatio, scilicet $p = \frac{dy}{dx}$, si enim ulterius progredi et insuper litteram q introducere velimus, ut sit

$$dp = qdx \quad \text{sive} \quad q = \frac{dp}{dx},$$

quaelibet formula differentialis tres adeo multiplicatores admitteret, quemadmodum ex forma simplicissima dq manifestum est, quae primum ipsa est integrabilis, seu multiplicator $= 1$, secundus multiplicator est y, quoniam

$$\int ydq = qy - \int qdy \quad \text{at} \quad \int qdy = \int pqdx = \int pdp = \frac{pp}{2},$$

unde fit

$$\int ydq = qy - \frac{pp}{2},$$

tertius vero multiplicator est x, quum sit

$$\int x dq = xq - p,$$

ex quo satis patet his casibus tria integrabilitatis genera locum habere.

14. Contemplemur autem hic tantum ternas variabilis x, y et p, existente $p = \frac{dy}{dx}$, et quo clarius appareat, semper duos multiplicatores primitivos locum habere, varios casus simpliciores in medium afferamus, quibus hos duos multiplicatores, sive divinando, sive quocunque alio modo, reperire licuit, quos casus sequenti modo adiungamus.

$$\text{I.}\ \alpha x dp + \beta p dx.$$

15. Haec formula primo per se est integrabilis, quum eius integrale sit

$$\alpha(px - y) + \beta y,$$

ita ut multiplicator primus sit $= 1$. Alter multiplicator erit $p^{\alpha-1}x^{\beta-1}$, tum enim integrale fit $p^\alpha x^\beta$. Pro multiplicatore igitur universali inveniendo erit ex § 10 $M = 1$ et $N = p^{\alpha-1}x^{\beta-1}$, hinc

$$v = \alpha(px - y) + \beta y = \alpha px + (\beta - \alpha)y \quad \text{et} \quad u = p^\alpha x^\beta,$$

quare si fuerit

$$dz = S dv + T du,$$

multiplicator generalis erit

$$S \cdot 1 + T p^{\alpha-1}x^{\beta-1}.$$

16. Unico autem casu, quo $\alpha = 1$ et $\beta = 1$, haec solutio fit incongrua, quippe quo ambo multiplicatores non amplius erunt diversi, uterque enim fieret $= 1$, hocque incommodum etiam usu venit, quoties $\beta = \alpha$, tum enim prius integrale est αpx et alter multiplicator $p^{\alpha-1}x^{\beta-1}$ eius foret potestas, neque propterea a priori multiplicatore differret, quod quidem per se evidens est, quum totum negotium a ratione inter α et β pendeat, quare hic nova quaestio oritur, num casu $\beta = \alpha$ exhiberi queat alius multiplicator, et quomodo futurus sit expressus, quem casum seorsim evolvamus.

II. $xdp + pdx$.

17. Circa priorem multiplicatorem $= 1$ hic nulla est difficultas, quum integrale sit px, alter vero multiplicator non tam facile se offert, re autem diligentius perpensa multiplicator se obtulit $= Lx$, erit enim

$$\int (xdp + pdx) Lx = pxLx - y,$$

simili autem modo alius colligitur multiplicator $\frac{y}{ppxx}$ [1]), fiet enim integrale

$$= \frac{-y}{px} + \int \frac{dy}{px} = \frac{-y}{px} + Lx,$$

neque vero hic multiplicator tertius a duobus prioribus discrepat, quum enim ex prioribus sit

$$M = 1, \; v = px \quad \text{et} \quad N = Lx, \quad u = pxLx - y,$$

manifestum est, tertium integrale esse functionem ipsius u et v, quum sit

$$\frac{u}{v} = Lx - \frac{y}{px}.$$

Ex hoc igitur exemplo intelligitur, saepenumero plures multiplicatores diversos videri posse, quum tamen ad duos reduci queant, ad quod diiudicandum tantum ex binis multiplicatoribus eliciantur litterae v et u, ex quibus semper reliqua integralia, quotcunque fuerint inventa, componi reperientur.

III. $\alpha y dp + \beta p dy$.

18. Hic iterum unus multiplicator sponte se offert, scilicet $p^{\alpha-1}y^{\beta-1}$, cui respondet integrale $p^\alpha y^\beta$ sive, quod eodem redit, sumto multiplicatore $\frac{1}{py}$, erit integrale

$$= \alpha Lp + \beta Ly = Lp^\alpha y^\beta,$$

quod quum fit illius logarithmus, etiam a priore differre non est censendum; alter multiplicator re probe perpensa colligitur $\frac{1}{\alpha} x y^{\frac{\beta-\alpha}{\alpha}}$ [2]), integrale enim erit

1) Editio princeps: $\frac{y}{pp + xy}$. Correxit H. D.

2) Editio princeps: $xpy^{\frac{\beta}{\alpha}}$. Correxit H. D.

$$xpy^{\frac{\beta}{\alpha}}-\left(\frac{\alpha}{\beta+\alpha}\right)y^{\frac{\beta+\alpha}{\alpha}},$$

tum etiam quasi sponte se prodit multiplicator $\frac{1}{pp}$, erit enim

$$\int\frac{\alpha y dp}{pp}=-\frac{\alpha y}{p}+\int\frac{\alpha dy}{p}=-\frac{\alpha y}{p}+\alpha x \quad \text{et} \quad \int\frac{pdy}{pp}=\int\frac{dy}{p}=x,$$

unde totum integrale erit

$$=\frac{-\alpha y}{p}+(\alpha+\beta)x,$$

hoc autem iam in duobus praecedentibus continetur, erit enim

$$M=p^{\alpha-1}y^{\beta-1} \quad \text{et} \quad v=p^{\alpha}y^{\beta},$$
$$N=xpy^{\frac{\beta}{\alpha}} \quad \text{et} \quad u=xpy^{\frac{\beta}{\alpha}}-\left(\frac{\alpha}{\beta+\alpha}\right)y^{\frac{\beta+\alpha}{\alpha}},$$

unde dividendo u per

$$\frac{v^{\frac{1}{\alpha}}}{\alpha+\beta}=\frac{py^{\frac{\beta}{\alpha}}}{\alpha+\beta}$$

oritur integrale $(\alpha+\beta)x-\frac{\alpha y}{p}$.

Haec autem reductio non succedit casu, quo $\beta=-\alpha$, quem casum seorsim evolvamus.

IV. $ydp-pdy$.

19. Cum hoc casu sit $\alpha=1$ et $\beta=-1$, erit prior multiplicator $\frac{1}{yy}$, praeterea vero colligitur multiplicator $\frac{x}{yy}$, tum enim erit

$$\int\frac{x(ydp-pdy)}{yy}=\frac{px}{y}-Ly,$$

haec enim formula differentiata praebet

$$\frac{x(ydp-pdy)}{yy}=\frac{pdx+xdp}{y}-\frac{dypx}{yy}-\frac{dy}{y} \quad \text{ob} \quad dy=pdx;$$

quum nunc habeamus duos multiplicatores, alterum $M = \frac{1}{yy}$ et alterum $N = \frac{x}{yy}$, unde fit

$$v = \frac{p}{y} \quad \text{et} \quad u = \frac{px}{y} - Ly\,,$$

si z denotet functionem quamcunque binarum quantitatum v et u, erit multiplicator generalis

$$= M\left(\frac{dz}{dv}\right) + N\left(\frac{dz}{du}\right) = \frac{1}{yy}\left(\frac{dz}{dv}\right) + \frac{x}{yy}\left(\frac{dz}{du}\right).$$

V. $pdp + xdx$.

20. Primo haec formula ipsa per se est integrabilis, ita ut sit

$$M = 1 \quad \text{et} \quad v = \tfrac{1}{2}(pp + xx),$$

tum vero alius multiplicator deprehenditur arcus, cuius tangens est $\frac{x}{p} = N$, tum enim erit

$$\int (pdp + xdx)\,\text{A. tang.}\frac{x}{p} = \tfrac{1}{2}(pp + xx)\,\text{A. tang.}\frac{x}{p} - \int \tfrac{1}{2}(pp + xx)\,d\cdot\text{A. tang.}\frac{x}{p}$$
$$= \tfrac{1}{2}(pp + xx)\,\text{A. tang.}\frac{x}{p} - \int \tfrac{1}{2}(pdx - xdp);$$

at est

$$\int \tfrac{1}{2}(pdx - xdp) = y - \frac{px}{2},$$

unde integrale quaesitum erit

$$\tfrac{1}{2}(pp + xx)\,\text{A. tang.}\frac{x}{p} - y + \frac{px}{2}.$$

21. Ex his exemplis abunde patet, inventionem huiusmodi multiplicatorum neutiquam esse obviam, sed saepenumero admodum esse absconditam, quin etiam evenire potest, ut vires analyseos plane superet. Interim tamen methodum quandam hic aperiam, ad hoc institutum accommodatam, cuius ope plurimis casibus tales multiplicatores invenire licet.

22. Quum ratio duplicis multiplicatoris in eo lateat, quod huiusmodi formulae differentiales sint secundi gradus, unde fit, ut uterque multiplicator unam tantum quasi integrationem involvat, ideoque duplex integratio etiam duplicem multiplicatorem requirat, hinc vicissim ambos multiplicatores reperire licebit, si utramque integrationem absolvamus. Quemadmodum igitur hac methodo uti oporteat, in sequentibus exemplis docebimus.

EXEMPLUM 1

23. Proposita formula differentiali, $xdp + pdx$, eius utrumque multiplicatorem invenire. Quum haec formula per se sit integrabilis, ideoque $M = 1$, ponatur

$$xdp + pdx = dv \quad \text{eritque} \quad px = v,$$

sicque una integratio est absoluta, pro altera vero, quum sit $p = \frac{v}{x}$, per dx multiplicando ob $pdx = dy$ habebimus $dy = \frac{vdx}{x}$, unde integrando elicimus

$$y = vLx - \int dv \cdot Lx, \quad \text{ideoque}$$

$$\int dvLx = vLx - y = pxLx - y,$$

unde intelligimus, formulam $dvLx$ esse integrabilem, siquidem eius integrale est $pxLx - y$, quare quum dv denotet ipsam formulam nostram propositam, $xdp + pdx$, patet eius multiplicatorem fore Lx.

24. Eodem modo etiam alios multiplicatores reperire licet, quum enim sit

$$dy = \frac{vdx}{x}, \quad \text{erit etiam} \quad \frac{dy}{v} = \frac{dx}{x},$$

hinc integrando

$$\frac{y}{v} + \int \frac{ydv}{vv} = Lx, \quad \text{ergo} \quad \int \frac{ydv}{vv} = Lx - \frac{y}{v} = Lx - \frac{y}{px},$$

integrabilis ergo est formula

$$\frac{ydv}{vv} \quad \text{seu} \quad dv . \frac{y}{vv},$$

sicque multiplicator erit

$$\frac{y}{vv} = \frac{y}{ppxx},$$

quem ergo loco N assumere licet, unde, quum sit

$$v = px \quad \text{et} \quad u = Lx - \frac{y}{px},$$

si z denotet functionem quamcunque ipsarum v et u, erit multiplicator generalis

$$\left(\frac{dz}{dv}\right) + \frac{y}{ppxx}\left(\frac{dz}{du}\right),$$

veluti si fuerit $z = vu$, erit

$$\left(\frac{dz}{dv}\right) = u \quad \text{et} \quad \left(\frac{dz}{du}\right) = v,$$

unde oritur hic multiplicator:

$$u + \frac{y}{ppxx}v = Lx - \frac{y}{px} + \frac{y}{px} = Lx,$$

qui est multiplicator priori loco inventus.

EXEMPLUM 2

25. Proposita formula differentiali

$$\alpha x dp + \beta p dx,$$

eius ambos multiplicatores invenire. Quum haec formula iam per se sit integrabilis, erit $M = 1$, et posito

$$\alpha x dp + \beta p dx = dv \quad \text{erit} \quad v = \alpha px + (\beta - \alpha)y,$$

unde colligitur

$$p = \frac{v}{\alpha x} + \frac{(\alpha - \beta)y}{\alpha x},$$

quae per dx multiplicata praebet

$$dy = p\,dx = \frac{v\,dx}{\alpha x} + \frac{\alpha-\beta}{\alpha}\cdot\frac{y\,dx}{x},$$

hincque

$$\frac{dx}{x} = \frac{\alpha\,dy}{v+(\alpha-\beta)y},$$

integrando igitur obtinebimus

$$Lx = \frac{\alpha}{\alpha-\beta}L[v+(\alpha-\beta)y] - \int\frac{\alpha}{\alpha-\beta}\cdot\frac{dv}{v+(\alpha-\beta)y},$$

sicque erit

$$\alpha\int\frac{dv}{v+(\alpha-\beta)y} = \alpha L[v+(\alpha-\beta)y] - (\alpha-\beta)Lx,$$

unde patet, formulae nostrae dv multiplicatorem fore

$$\frac{\alpha}{v+(\alpha-\beta)y} = \frac{1}{px},$$

uti per se est manifestum, tum enim integrale erit

$$\alpha Lp + \beta Lx.$$

EXEMPLUM 3

26. Proposita formula $p\,dp + x\,dx$, eius ambos multiplicatores invenire. Hic iterum primus multiplicator est $M = 1$ et posito

$$p\,dp + x\,dx = dv \quad \text{erit} \quad pp + xx = 2v, \quad \text{hinc} \quad p = \sqrt{(2v - xx)}$$

et per dx multiplicando

$$dy = p\,dx = dx\sqrt{(2v - xx)},$$

ponatur tantisper $2v = ss$, fietque

$$y = \tfrac{1}{2}x\cdot\sqrt{(ss-xx)} + \frac{ss}{2}\text{Arc. sin.}\frac{x}{s} - \int\left(\frac{xs\,ds}{2\sqrt{(ss-xx)}} + s\,ds\,\text{Arc. sin.}\frac{x}{s} - \frac{sx\,ds}{2\sqrt{(ss-xx)}}\right)$$
$$= \tfrac{1}{2}x\cdot\sqrt{(ss-xx)} + \frac{ss}{2}\text{Arc. sin.}\frac{x}{s} - \int s\,ds\,\text{Arc. sin.}\frac{x}{s},$$

ideoque

$$\int s\,ds \cdot \text{A. sin.}\frac{x}{s} = \tfrac{1}{2}x\sqrt{(ss - xx)} + \frac{ss}{2}\,\text{A. sin.}\frac{x}{s} - y$$
$$= \frac{px}{2} + \tfrac{1}{2}(pp + xx)\ \text{A. sin.}\ \frac{x}{\sqrt{(pp + xx)}} - y\,^{1)}$$

at est

$$s\,ds = p\,dp + x\,dx,$$

unde patet, nostrae formulae multiplicatorem esse

$$\text{Arc. sin.}\frac{x}{\sqrt{(pp + xx)}}\,,\quad \text{sive}\quad \text{Arc. tang.}\frac{x}{p}\,.$$

27. Haec autem operatio nimis est molesta, quam ut ea in formulis magis complicatis uti queamus, quare eam sequenti modo faciliorem reddere conemur. Quum invenerimus

$$dy = dx\sqrt{(ss - xx)}\,,$$

statuamus hic $x = sz$ fietque

$$dy = ss\,dz\sqrt{(1 - zz)} + zs\,ds\sqrt{(1 - zz)}\,,$$

quae per ss divisa dat

$$\frac{dy}{ss} = dz\sqrt{(1 - zz)} + \frac{z\,ds}{s}\sqrt{(1 - zz)}$$

et integrando

$$\frac{y}{ss} + 2\int\frac{y\,ds}{s^3} = \int dz\sqrt{(1 - zz)} + \int\frac{z\,ds}{s}\sqrt{(1 - zz)}\,,$$

ubi membrum penultimum

$$\int dz\sqrt{(1 - zz)}$$

absolute datur, quum sit certa quaedam functio ipsius $z = \frac{x}{s}$, postremum autem membrum restituto pro z valore $\frac{x}{s}$ abit in

$$\int\frac{x\,ds}{s^3}\sqrt{(ss - xx)}\,,$$

unde binis integralibus per $\frac{ds}{s^3}$ affectis coniungendis consequimur

1) A. sin $\frac{x}{s}$ et infra Ar. sin $\frac{x}{\sqrt{(pp + xx)}}$ idem significant quod Arc. sin. $\frac{x}{s}$. H. D.

$$\int \frac{ds}{s^3}(2y - x\sqrt{(ss-xx)}) = \int dz\sqrt{(1-zz)} - \frac{y}{ss},$$

ex quo patet, formulae nostrae

$$pdp + xdx = sds$$

multiplicatorem esse

$$\frac{1}{s^4}(2y - x\sqrt{(ss-xx)}),$$

qui ob $ss = pp + xx$ transmutatur in hanc formam:

$$\frac{2y-px}{(pp+xx)^2},$$

qui si ponatur $= N$, erit

$$u = \int dz\sqrt{(1-zz)} - \frac{y}{pp+xx} \text{ existente } z = \frac{x}{\sqrt{(pp+xx)}}$$

atque hinc multiplicatorem generalem facile elicere licet.

28. Hinc patet, si formula proposita sit $\alpha pdp + \beta xdx$, quo casu iterum est

$$M = 1 \quad \text{et} \quad v = \frac{\alpha pp}{2} + \frac{\beta xx}{2},$$

alterum multiplicatorem repertum iri

$$N = \frac{2y - xp}{(\alpha pp + \beta xx)^2},$$

tum enim integrale hinc natum erit

$$u = \int \frac{sds}{s^4}(2y - px) = \int dz\sqrt{\frac{1-\beta zz}{\alpha}} - \frac{y}{\alpha pp + \beta xx},$$

existente

$$z = \frac{x}{s} = \frac{x}{\sqrt{(\alpha pp + \beta xx)}}.$$

EXEMPLUM 4

29. Proposita formula differentiali

$$p^{n-1}dp + \beta x^{n-1}dx,$$

eius multiplicatores invenire. Quum alter multiplicator iterum sit cognitus $= 1$, posita nostra formula $= dv$, erit

$$p^n + \beta x^n = nv,$$

unde fit

$$p = (nv - \beta x^n)^{\frac{1}{n}},$$

hincque

$$dy = pdx = dx(nv - \beta x^n)^{\frac{1}{n}},$$

statuatur nunc

$$nv = s^n, \quad \text{sitque} \quad x = sz,$$

habebimus

$$dy = (sdz + zds)s(1 - \beta z^n)^{\frac{1}{n}},$$

quae per ss divisa dat

$$\frac{dy}{ss} = dz(1 - \beta z^n)^{\frac{1}{n}} + \frac{zds}{s}(1 - \beta z^n)^{\frac{1}{n}},$$

hinc integrando

$$\frac{y}{ss} + \int \frac{2yds}{s^3} = \int dz(1 - \beta z^n)^{\frac{1}{n}} + \int \frac{zds}{s}(1 - \beta z^n)^{\frac{1}{n}},$$

ubi membrum penultimum est determinatum, quippe certa quaedam functio ipsius

$$z = \frac{x}{s} = \frac{x}{(nv)^{\frac{1}{n}}},$$

ultimum vero membrum, si in eo restituatur $z = \frac{x}{s}$, abit in

$$\int \frac{xds}{s^3}(s^n - \beta x^n)^{\frac{1}{n}},$$

atque ob

$$(s^n - \beta x^n)^{\frac{1}{n}} = p$$

erit

$$\int \frac{xds}{s^3}(s^n - \beta x^n)^{\frac{1}{n}} = \int \frac{pxds}{s^3},$$

quibus substitutis fiet

$$\int \frac{ds}{s^3}(2y - px) = \int dz(1 - \beta z^n)^{\frac{1}{n}} - \frac{y}{ss}.$$

Quum autem sit $s = (nv)^{\frac{1}{n}}$, erit

$$ds = (nv)^{\frac{1}{n}-1}dv \quad \text{et} \quad \frac{ds}{s^3} = \frac{dv}{(nv)^{\frac{2+n}{n}}},$$

ex quo primum membrum fiet

$$\int \frac{dv}{(nv)^{\frac{2+n}{n}}}(2y - px),$$

cuius integrale quum iam sit inventum, patet formulae propositae dv multiplicatorem esse

$$= \frac{2y - px}{(nv)^{\frac{2+n}{n}}} = \frac{2y - px}{(p^n + \beta x^n)^{\frac{2+n}{n}}}.$$

EXEMPLUM 5

30. Proposita formula differentiali:

$$p^{m-1}dp + \beta x^{n-1}dx = dv,$$

eius multiplicatorem alterum invenire. Quum hinc sit

$$v = \frac{1}{m}p^m + \frac{\beta}{n}x^n, \quad \text{erit}$$

$$p = (mv - \frac{m\beta}{n}\cdot x^n)^{\frac{1}{m}} \quad \text{et} \quad dy = pdx = dx(mv - \frac{m\beta}{n}\cdot x^n)^{\frac{1}{m}};$$

ponatur hic iterum

$$mv = s^n \quad \text{et} \quad x = sz, \quad \text{erit}$$

$$dy = (sdz + zds)s^{\frac{n}{m}}\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}},$$

quae divisa per $s^{\frac{m+n}{m}}$ praebet

$$\frac{dy}{s^{\frac{m+n}{m}}} = dz(1 - \frac{m\beta}{n}\cdot z^n)^{\frac{1}{m}} + \frac{zds}{s}(1 - \frac{m\beta}{n}\cdot z^n)^{\frac{1}{m}},$$

hinc integrando

$$\frac{y}{s^{\frac{m+n}{m}}} + \int \frac{m+n}{m}\cdot\frac{yds}{s^{\frac{2m+n}{m}}} = \int dz(1 - \frac{m\beta}{n}\cdot z^n)^{\frac{1}{m}} + \int \frac{zds}{s}(1 - \frac{m\beta}{n}\cdot z^n)^{\frac{1}{m}},$$

ubi membrum penultimum est functio cognita ipsius z, ultimum autem substituto loco z valore $\frac{x}{s}$, abit in

$$\int \frac{xds}{s^{\frac{2m+n}{m}}}\left(s^n - \frac{m\beta}{n}\cdot x^n\right)^{\frac{1}{m}} = \int \frac{pxds}{s^{\frac{2m+n}{m}}},$$

quare hinc colligimus

$$\int \frac{ds}{s^{\frac{2m+n}{m}}}\left(\frac{m+n}{m}y - px\right) = \int dz\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}} - \frac{y}{s^{\frac{m+n}{m}}},$$

quum autem sit

$$s = (mv)^{\frac{1}{n}}, \quad \text{erit} \quad ds = \frac{1}{n}\cdot m^{\frac{1}{n}}\, v^{\frac{1}{n}-1}dv$$

et

$$\frac{ds}{s^{\frac{2m+n}{m}}} = \frac{1}{n}\cdot\frac{dv}{m^{\frac{m+n}{mn}}\, v^{\frac{m+n}{mn}+1}},$$

ex quo colligitur formulae nostrae propositae multiplicatorem esse

$$\frac{\frac{m+n}{m}y - px}{v^{\frac{m+n}{mn}+1}} = \frac{\frac{m+n}{m}y - px}{\left(\frac{1}{m}p^m + \frac{\beta}{n}x^n\right)^{\frac{m+n}{mn}+1}}.$$

31. Hactenus eiusmodi formulas sumus contemplati, quae cum per se sunt integrabiles, tum vero duas tantum variabiles p et x contineant; simili autem modo istas formulas tractare licebit, quae tantum has duas variabiles p et y involvant, ubi quidem assumimus, has formulas per se esse integrabiles.

EXEMPLUM 6

Proposita formula

$$p^{m-1}dp + \beta y^{n-1}dy = dv,$$

eius alterum multiplicatorem investigare. Primum integrando colligimus

$$v = \frac{1}{m}p^m + \frac{\beta}{n}y^n,$$

unde fit

$$p = \left(mv - \frac{m\beta}{n}\cdot y^n\right)^{\frac{1}{m}}$$

et

$$dy = p\,dx = dx\left(mv - \frac{m\beta}{n}\cdot y^n\right)^{\frac{1}{m}}$$

ideoque

$$dx = \frac{dy}{\left(mv - \frac{m\beta}{n}\cdot y^n\right)^{\frac{1}{m}}};$$

statuatur iam hic $mv = s^n$ et $y = sz$, ut obtineamus

$$dx = \frac{s\,dz + z\,ds}{s^{\frac{n}{m}}\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}}},$$

quae aequatio divisa per $s^{\frac{m-n}{m}}$ dat

$$\frac{dx}{s^{\frac{m-n}{m}}} = \frac{dz}{\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}}} + \frac{z\,ds}{s\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}}};$$

haec aequatio simili modo ut supra integratur, et ex penultimo membro iterum nascitur functio determinata ipsius z; si in membro postremo loco z ipsius valorem $\frac{y}{s}$ restituamus, consequimur

$$\frac{x}{s^{\frac{m-n}{m}}} + \int \frac{m-n}{m}\cdot\frac{x\,ds}{s^{\frac{2m-n}{m}}} = \int\frac{dz}{\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}}} + \int\frac{y\,ds}{p\cdot s^{\frac{2m-n}{m}}}\,{}^{1)},$$

ex his igitur colligimus

$$\int\frac{ds}{s^{2-\frac{n}{m}}}\left(\left(1 - \frac{n}{m}\right)x - \frac{y}{p}\right) = \int\frac{dz}{\left(1 - \frac{m\beta}{n}\cdot z^n\right)^{\frac{1}{m}}} - \frac{x}{s^{\frac{m-n}{m}}},$$

quum nunc sit $s = (mv)^{\frac{1}{n}}$, erit

1) Editio princeps: $\int\frac{y\,ds}{s^{\frac{2m-n}{n}}}$. Correxit H. D.

$$ds = \frac{1}{n} \cdot m^{\frac{1}{n}} v^{\frac{1}{n}-1} dv\,,$$

hinc

$$\frac{ds}{s^{2-\frac{n}{m}}} = \frac{1}{n} dv \cdot \frac{1}{m^{\frac{1}{n}-\frac{1}{m}} v^{\frac{1}{n}-\frac{1}{m}+1}}\,,$$

unde concludimus formulae nostrae propositae multiplicatorem fore

$$\frac{\left(1-\frac{n}{m}\right)x-\frac{y}{p}}{v^{\frac{1}{n}-\frac{1}{m}+1}} = \frac{\left(1-\frac{n}{m}\right)x-\frac{y}{p}}{\left(\frac{1}{m}p^m+\frac{\beta}{n}y^n\right)^{\frac{1}{n}-\frac{1}{m}+1}}\,.$$

32. Etsi haec exempla iam satis late patere videntur, tamen, si res ipsa spectetur, ea etiam nunc sunt vehementer particularia, siquidem pro v formula binomialis utroque casu prodit, in exemplo penultimo litteras p et x, in ultimo vero p et y involvens, atque hinc vix liquet, quomodo operationes institui oporteat, si plures termini in valore ipsius v occurrant. Interim tamen sequenti modo ista investigatio multo generalior reddi poterit.

PROBLEMA

Si Ω eiusmodi fuerit functio quantitatum p et x, ut ea posito $p = x^\lambda q$ induat hanc formam $x^n Q$, ita ut Q tantum sit functio ipsius q, tum proposita formula differentiali $d\Omega$ eius alterum multiplicatorem invenire.

SOLUTIO

Posito ut ante $d\Omega = dv$, ut sit $v = \Omega$, ponatur $p = x^\lambda q$ eritque per hypothesin

$$v = x^n Q\,, \quad \text{hinc} \quad Q = \frac{v}{x^n}\,,$$

nunc statuatur porro $x^n = \frac{v}{z}$, ut fiat $Q = z$, iam quotcunque dimensiones ipsius q in functione Q contineantur, etiamsi resolutio istius aequationis vires analyseos superet, tamen certum est, inde valorem radicis q per certam quandam functionem ipsius z, quae sit Z, expressum iri, ita ut sit $q = Z$, hinc

$$p = x^\lambda Z \quad \text{et} \quad dy = p\,dx = x^\lambda Z\,dx,$$

iam cum sit

$$x^n = \frac{v}{z}, \quad \text{erit} \quad x^{\lambda+1} = \left(\frac{v}{z}\right)^{\frac{\lambda+1}{n}}$$

et

$$x^\lambda dx = \frac{1}{n}\cdot\frac{z\,dv - v\,dz}{zz}\left(\frac{v}{z}\right)^{\frac{\lambda+1}{n}-1} = \frac{1}{n}\left(\frac{dv\cdot v^{\frac{\lambda+1}{n}-1}}{z^{\frac{\lambda+1}{n}}} - \frac{dz\cdot v^{\frac{\lambda+1}{n}}}{z^{\frac{\lambda+1}{n}+1}}\right)$$

et multiplicando per $\frac{n}{v^{\frac{\lambda+1}{n}}}$ habebimus

$$\frac{n\,dy}{v^{\frac{\lambda+1}{n}}} = \frac{Z\,dv}{v\cdot z^{\frac{\lambda+1}{n}}} - \frac{Z\,dz}{z^{\frac{\lambda+1}{n}+1}},$$

ubi membrum ultimum variabilem z continens dabit functionem determinatam ipsius z, penultimum vero pro z restituto valore $\frac{v}{x^n}$ ob $Z = \frac{p}{x^\lambda}$ abit in $\frac{px\,dv}{v^{\frac{\lambda+1}{n}+1}}$, quare integrando habebimus

$$\frac{ny}{v^{\frac{\lambda+1}{n}}} + \int(\lambda+1)\frac{y\,dv}{v^{\frac{\lambda+1}{n}+1}} = \int\frac{px\,dv}{v^{\frac{\lambda+1}{n}+1}} - \int\frac{Z\,dz}{z^{\frac{\lambda+1}{n}+1}},$$

unde fit

$$\int\frac{dv}{v^{\frac{\lambda+1}{n}+1}}(px-(\lambda+1)y) = \int\frac{Z\,dz}{z^{\frac{\lambda+1}{n}+1}} + \frac{ny}{v^{\frac{\lambda+1}{n}}}.$$

Quocirca concludimus, formulae nostrae $dv = d\Omega$ multiplicatorem quaesitum esse

$$\frac{px-(\lambda+1)y}{(\Omega)^{\frac{\lambda+1}{n}+1}}.$$

PROBLEMA

33. Si Ω eiusmodi fuerit functio ipsarum p et y, ut posito $p = y^\lambda q$ ea obtineat hanc formam $y^n Q$, existente Q functione ipsius q, tum proposita formula differentiali $d\Omega$ eius multiplicatorem invenire.

SOLUTIO

Posito iterum $d\Omega = dv$ erit $v = \Omega$, et posito $p = y^\lambda q$ fiet $v = y^n Q$, hincque $Q = \frac{v}{y^n}$; statuatur nunc $y^n = \frac{v}{z}$, ut fiat $Q = z$, unde quum Q sit functio ipsius q, per resolutionem aequationum q aequabitur certae cuipiam functioni ipsius z, quae sit Z, ita ut sit

$$q = Z, \quad \text{hinc} \quad p = y^\lambda Z$$

$$\text{et } dy = pdx = y^\lambda Z dx\,, \quad \text{unde fit } dx = \frac{dy}{y^\lambda Z},$$

quum autem sit $y^n = \frac{v}{z}$, erit

$$y = \frac{v^{\frac{1}{n}}}{z^{\frac{1}{n}}} \text{ et } dy = \frac{1}{n}\left(\frac{dv \cdot v^{\frac{1}{n}-1}}{z^{\frac{1}{n}}} - \frac{v^{\frac{1}{n}} dz}{z^{\frac{1}{n}+1}}\right),$$

ex quo habebitur

$$dx = \frac{1}{n} \frac{v^{\frac{1}{n}-\frac{\lambda}{n}-1} dv}{z^{\frac{1-\lambda}{n}} \cdot Z} - \frac{1}{n} \cdot \frac{v^{\frac{1-\lambda}{n}} dz}{Z \cdot z^{\frac{1-\lambda}{n}+1}},$$

quod multiplicatum per $n \cdot v^{\frac{\lambda-1}{n}}$ praebebit

$$n dx \cdot v^{\frac{\lambda-1}{n}} = \frac{dv}{v \cdot z^{\frac{1-\lambda}{n}} \cdot Z} - \frac{dz}{z^{\frac{1-\lambda}{n}+1} \cdot Z};$$

hic ultimi membri integrale manifesto est certa potestas ipsius z, quam saltem per quadraturas exhibere licet; penultimum vero membrum ob

$$Z = q = \frac{p}{y^\lambda} \text{ abit in } \frac{dv y^\lambda}{pv \cdot z^{\frac{1-\lambda}{n}}}$$

et substituto pro z eius valore $\frac{v}{y^n}$ transformatur in

$$\frac{y dv}{p \cdot v^{\frac{1-\lambda}{n}+1}},$$

quare primo membro per reductionem integrando consequimur

$$nx \cdot v^{\frac{\lambda-1}{n}} - \int (\lambda - 1)\, x \cdot v^{\frac{\lambda-1}{n}-1} dv = \int \frac{y\, dv \cdot v^{\frac{\lambda-1}{n}-1}}{p} - \int \frac{dz \cdot z^{\frac{\lambda-1}{n}-1}}{Z},$$

atque hinc colligitur

$$\int dv \cdot v^{\frac{\lambda-1}{n}-1} \left((\lambda - 1)\, x + \frac{y}{p}\right) = nx \cdot v^{\frac{\lambda-1}{n}} + \int \frac{dz \cdot z^{\frac{\lambda-1}{n}-1}}{Z},$$

quum igitur sit $dv = d\Omega$, patet formulam propositam $d\Omega$ integrabilem reddi, si multiplicetur per

$$(\Omega)^{\frac{\lambda-1}{n}-1} \left((\lambda - 1)\, x + \frac{y}{p}\right),$$

qui idcirco est multiplicator quaesitus.

34. Quum in problemate penultimo formulae $d\Omega$ multiplicator sit

$$\frac{px - (\lambda + 1)\, y}{(\Omega)^{\frac{\lambda+1}{n}+1}},$$

huius formulae

$$\frac{d\Omega}{(\Omega)^{\frac{\lambda+1}{n}+1}},$$

quae etiam est verum differentiale, quippe cuius integrale est

$$\frac{n}{\lambda + 1} \cdot \frac{1}{(\Omega)^{\frac{\lambda+1}{n}}},$$

multiplicator erit

$$px - (\lambda + 1)\, y;$$

simili modo quum in ultimo problemate formulae $d\Omega$ multiplicator sit inventus

$$(\Omega)^{\frac{\lambda-1}{n}-1} \left((\lambda - 1)\, x + \frac{y}{p}\right),$$

erit huius formulae

$$d\Omega\, (\Omega)^{\frac{\lambda-1}{n}-1},$$

quae etiam est verum differentiale, quippe cuius integrale est

$$\frac{n}{\lambda - 1} \cdot (\Omega)^{\frac{\lambda-1}{n}},$$

multiplicator

$$(\lambda - 1)x + \frac{y}{p};$$

hi duo casus ob simplicitatem multiplicatoris inprimis notatu digni videntur, ex quo operae pretium erit, in genere omnes formulas differentiales investigare, quibus talis multiplicator conveniat, quem in finem hoc Lemma praemittimus.

LEMMA

35. Si formulae differentialis $d\Omega$ multiplicator fuerit V, tum vicissim formulae differentialis dV multiplicator erit Ω; quum enim sit

$$\int \Omega\, dV = V\Omega - \int V d\Omega,$$

quoniam formula $Vd\Omega$ per hypothesin est integrabilis, necesse est, etiam formulam $\int \Omega\, dV$ esse integrabilem.

PROBLEMA

36. Invenire omnes formulas differentiales $d\Omega$, quibus conveniat multiplicator

$$\alpha y + px,$$

denotante α numerum quemcunque.

SOLUTIO

Quum ob $dy = pdx$ sit

$$d(\alpha y + px) = (\alpha + 1)pdx + xdp,$$

huius formulae multiplicatorem oportet esse Ω, ex qua conditione quantitatem Ω determinare licet. Sit igitur proposita haec formula differentialis

$$(\alpha + 1)pdx + xdp,$$

pro qua habetur multiplicator

$$M = 1 \quad \text{eritque} \quad v = \alpha y + px,$$

alter vero multiplicator erit

$$N = x^\alpha, \quad \text{fietque tum} \quad u = p \cdot x^{\alpha+1},$$

quare si Z denotet functionem quamcunque binarum variabilium

$$v = \alpha y + px \quad \text{et} \quad u = p \cdot x^{\alpha+1},$$

in genere multiplicator nostrae formulae erit

$$M\left(\frac{dZ}{dv}\right) + N\left(\frac{dZ}{du}\right),$$

quamobrem, quum sit

$$\Omega = \left(\frac{dZ}{dv}\right) + x^\alpha\left(\frac{dZ}{du}\right),$$

eius differentiale $d\Omega$ continebit omnes formulas differentiales, quibus convenit multiplicator $\alpha y + px$.

37. Si pro Z sumatur functio quaecunque ipsius u, erit

$$\left(\frac{dZ}{dv}\right) = 0$$

et $\left(\frac{dZ}{du}\right)$ erit functio quaedam ipsius u, quae sit $f:u$[1]), hinc nostrum erit Ω

$$x^\alpha \cdot f:u = x^\alpha f:p \cdot x^{\alpha+1},$$

quae forma probe convenit cum problemate § 32 [vide quoque § 34], ubi multiplicator erat $px - (\lambda + 1)y$, ita ut sit $\alpha = -(\lambda + 1)$. Deinde quod hic est $d\Omega$,

$$\text{ibi erat } \frac{dv}{v^{\frac{\lambda+1}{n}+1}},$$

sicque quod hic est Ω,

$$\text{ibi erat } \frac{n}{\lambda+1} \cdot \frac{1}{v^{\frac{\lambda+1}{n}}},$$

at vero ibi erat

$$v = x^n Q = x^n \Phi : \frac{p}{x^\lambda},$$

ita ut inde sit

$$\frac{1}{v^{\frac{\lambda+1}{n}}} = x^{-(\lambda+1)} \cdot \Delta : \frac{p}{x^\lambda},$$

1) Hic et in sequentibus formulis $f:u$ denotat functionem ipsius u. H. D.

quam formam in illa contineri, satis manifestum est. Hinc ergo patet illud problema esse casum maxime particularem huius problematis, huiusque solutionem infinities latius patere.

PROBLEMA

38. Invenire omnes formulas differentiales $d\Omega$, quibus conveniat multiplicator $\alpha x+\frac{y}{p}$.

SOLUTIO

Quum ob $dx=\frac{dy}{p}$ sit

$$d\cdot\left(\alpha x+\frac{y}{p}\right)=(\alpha+1)\frac{dy}{p}-\frac{y\,dp}{pp},$$

consideremus hoc ipsum differentiale tanquam formulam propositam, cuius multiplicator Ω sit investigandus, et quum primus multiplicator sit $M=1$, erit $v=\alpha x+\frac{y}{p}$, alter autem multiplicator colligitur $N=y^{\alpha}$, ex quo fit $u=\frac{y^{\alpha+1}}{p}$, quare si Z denotet functionem quamcunque binarum variabilium

$$v=\alpha x+\frac{y}{p} \quad\text{et}\quad u=\frac{y^{\alpha+1}}{p},$$

expressio generalis pro multiplicatore quaesito erit

$$\Omega=\left(\frac{dZ}{dv}\right)+y^{\alpha}\left(\frac{dZ}{du}\right),$$

ubi notandum, si pro Z sumatur tantum functio unicae variabilis u, tum hanc solutionem ad casum problematis § 33 perducere.

39. Quae hactenus de investigatione multiplicatorum sunt tradita, insignem praestant usum in resolutione aequationum differentialium secundi gradus, quum enim, ob $dy=p\,dx$, omnes aequationes huius ordinis ad hanc formam redigere liceat:

$$P\,dp+Q\,dx+R\,dy=0,$$

manifestum est, si unus huius formulae multiplicator innotescat, tum statim aequationem semel integratam, quae erit differentialis primi ordinis, obtineri, quam deinceps secundum praecepta cognita tractari conveniet; at vero si bini

eius formulae multiplicatores fuerint cogniti, tum statim aequationem finitam seu bis integratam elicere licebit, ita ut non opus sit integratione repetita, quam operationem sequenti problemate doceamus.

PROBLEMA

40. Proposita aequatione differentiali

$$Pdp + Qdx + Rdy = 0,$$

si eius duo multiplicatores innotescant M et N, eius aequationem finitam bis integratam invenire.

SOLUTIO

Quum M et N sint multiplicatores cogniti, ponamus

$$M(Pdp + Qdx + Rdy) = dv$$

et

$$N(Pdp + Qdx + Rdy) = du,$$

hinc habebuntur quantitates v et u, quomodocunque ex ternis variabilibus p, x et y conflatae. Ob priorem igitur multiplicatorem fiet $v = a$ et ob posteriorem $u = b$, ubi a et b sunt binae constantes, utraque integratione ingressae; quum igitur duae habeantur aequationes finitae inter ternas variabiles x, y et p, si inde p eliminetur, prodibit aequatio finita inter binas coordinatas x et y, vel quod eodem redit, inde determinari poterunt binae harum litterarum per tertiam.

41. Hanc methodum ex praecedentibus per plura exempla facile illustrare liceret, verum instar omnium nobis erit problema initio huius dissertationis commemoratum, quo linea curva aequatione inter binas coordinatas x et y exprimenda quaeritur, cuius radius osculi aequetur formulae $\frac{1}{n}\sqrt{(xx + yy)}$.

EXEMPLUM

Proposita aequatione differentiali secundi gradus

$$\frac{dp}{(1 + pp)^{\frac{3}{2}}} - \frac{ndx}{\sqrt{(xx + yy)}} = 0,$$

invenire aequationem finitam inter x et y, ope duplicis multiplicatoris. Primum multiplicatorem iam supra observavimus [§ 6] esse $x + py$, unde fit

$$dv = \frac{dp(x + py)}{(1 + pp)^{\frac{3}{2}}} - \frac{n(xdx + ydy)}{\sqrt{(xx + yy)}},$$

adeoque erit

$$v = \frac{px - y}{\sqrt{(1 + pp)}} - n\sqrt{(xx + yy)} = a.$$

Pro altero multiplicatore, quoniam is tam facile non liquet, tam eum, quam integrale inde natum simul investigemus, ponamus primo $y = xz$, et ob $dy = pdx$ fiet

$$pdx = zdx + xdz \quad \text{hincque} \quad \frac{dx}{x} = \frac{dz}{p - z},$$

quo valore substituto, formula nostra erit

$$\frac{dp}{(1 + pp)^{\frac{3}{2}}} - \frac{ndz}{(p - z)\sqrt{(1 + zz)}} = 0,$$

quare posito porro $z = \dfrac{p + q}{1 - pq}$, unde fit

$$p - z = -\frac{q(1 + pp)}{1 - pq}, \quad \sqrt{(1 + zz)} = \frac{\sqrt{((1 + pp)(1 + qq))}}{1 - pq}$$

et

$$dz = \frac{dp(1 + qq) + dq(1 + pp)}{(1 - pq)^2},$$

transformatur nostra formula in hanc:

$$\frac{dp}{(1 + pp)^{\frac{3}{2}}} + \frac{ndp(1 + qq) + ndq(1 + pp)}{q(1 + pp)^{\frac{3}{2}}\sqrt{(1 + qq)}} = \frac{(q\sqrt{(1 + qq)} + n(1 + qq))dp + ndq(1 + pp)}{q(1 + pp)^{\frac{3}{2}}\sqrt{(1 + qq)}},$$

vel in

$$\frac{q + n\sqrt{(1 + qq)}}{q\sqrt{(1 + pp)}}\left(\frac{dp}{1 + pp} + \frac{ndq}{q\sqrt{(1 + qq)} + n(1 + qq)}\right)^{1)},$$

hinc igitur patet, pro altero multiplicatore sumi debere

1) Editio princeps in hac formula et in sequentibus habet $q + n\sqrt{(1 + qq)}$ loco $q\sqrt{(1 + qq)} + n(1 + qq)$. Correxit H. D.

$$N = \frac{q\sqrt{(1+pp)}}{q+n\sqrt{(1+qq)}},$$

tumque fit

$$du = \frac{dp}{1+pp} + \frac{n\,dq}{q\sqrt{(1+qq)}+n(1+qq)},$$

quae formula est integrabilis fietque inde

$$u = \int\frac{dp}{1+pp} + \int\frac{n\,dq}{q\sqrt{(1+qq)}+n(1+qq)} = b,$$

ex qua vel q per p, vel p per q determinabitur, tum vero, quum sit

$$z = \frac{p+q}{1-pq} = \frac{y}{x},$$

inde fit

$$y = \frac{(p+q)x}{1-pq},$$

qui valor in prima aequatione substitutus praebet

$$\frac{-qx\sqrt{(1+pp)}}{1-pq} - nx\frac{\sqrt{((1+pp)(1+qq))}}{1-pq} = a,$$

nunc igitur inveniri potest x, unde colligitur

$$x = \frac{-a(1-pq)}{(q+n\sqrt{(1+qq)})\sqrt{(1+pp)}},$$

sicque quia datur p per q, vel q per p, etiam x eodem modo definitur, deinde vero erit

$$y = \frac{-a(p+q)}{(q+n\sqrt{(1+qq)})\sqrt{(1+pp)}},$$

quae est solutio completa problematis.

OBSERVATIONES CIRCA AEQUATIONEM DIFFERENTIALEM $y\,dy + My\,dx + N\,dx = 0$

Commentatio 430 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae 17 (1772), 1773, p. 105—124
Summarium ibidem p. 11—12

SUMMARIUM

Aequatio proposita, in qua evolvenda Ill. EULERUS hic versatur, quanquam in se satis simplex videatur, eius tamen integratio tam est intricata, ut omnes adhuc Analyseos vires eluserit. Continetur ea sub forma generaliori, olim a Comite RICCATIO tractata:

$$dz + Pz^n dx + Q\,dx = 0,$$

ad quam duplici modo revocatur, primo si per y dividatur, deinde etiam ponendo $yy = z$; priori enim casu prodit

$$dy + M\,dx + \frac{N\,dx}{y} = 0,$$

ita ut sit $n = -1$; posteriori vero positione fit

$$dz + 2Mz^{\frac{1}{2}}dx + 2N\,dx = 0,$$

ut sit $n = \frac{1}{2}$. In tractanda hac aequatione Ill. Auctor methodum generalem dudum a se expositam in auxilium vocat, qua ostendit, omnium aequationum differentialium integrationem commodissime per multiplicatores absolvi; unde Cel. Vir id operam dat, ut binarum variabilium x et y investiget functionem eiusmodi z, per quam aequatio proposita divisa reddatur integrabilis.

1. In hac aequatione quantitates M et N ut functiones quaecunque variabilis x spectantur; et cum haec aequatio ita sit comparata, ut in genere

nullo modo integrari queat, methodis saltem etiamnunc cognitis, praecipua quaestio in ea indole binarum functionum M et N investiganda versatur, qua integratio absolvi queat; unde quidem casus per se obvios, veluti quando vel altera evanescit, vel ambae in ratione constante consistunt, excludi convenit.

2. Cum forma huius aequationis satis sit simplex, ut mirari liceat, quod eius integratio vires Analyseos adhuc eluserit, eius certe consideratio eo maiori attentione digna videtur; idque potissimum quod in forma generaliori, a Comite RICCATIO olim tractata

$$dz + Pz^n dx + Q dx = 0$$

continetur, ad quam adeo duplici modo revocari potest. Primo enim per y divisa praebet

$$dy + Ny^{-1} dx + M dx = 0,$$

ita ut $n = -1$, tum vero posito $yy = z$, prodit

$$dz + 2Mz^{\frac{1}{2}} dx + 2N dx = 0,$$

ita ut sit $n = \frac{1}{2}$. Ubi in genere observasse iuvabit, posito $z = y^{\frac{-1}{n-1}}$ formam generalem in hanc mutari:

$$dy + (1-n) Q y^{\frac{n}{n-1}} dx + (1-n) P dx = 0,$$

ita ut in hoc negotio exponentes n et $\frac{n}{n-1}$ pro aequivalentibus sint habendi.

3. Patet ergo, quod tantum in transitu monuerim, casum $n = 2$, quem RICCATIUS olim imprimis est contemplatus, hac proprietate prae reliquis esse praeditum, ut hac reductione ad se ipsum revocetur, dum $\frac{n}{n-1}$ iterum dat binarium. Tum vero casus $n = 1$, quo aequatio

$$dz + Pz dx + Q dx = 0$$

generatim est integrabilis, perducitur ad alterum $\frac{n}{n-1} = \infty$, unde cum potestati y^∞ aequivaleat quasi forma exponentialis e^y, etiam haec aequatio

$$dy + Pe^y dx + Q dx = 0$$

pro integrabili est habenda, cuius integratio ponendo

$$e^y = \frac{1}{v},$$

ut fiat

$$y = -lv \quad \text{et} \quad dy = -\frac{dv}{v},$$

hincque prodeat

$$dv - Qvdx - Pdx = 0,$$

per se est manifesta.

4. Ut autem ad ipsam aequationem propositam revertar, ante omnia observo me ad similem formam, cum olim aequationem differentialem tertii gradus[1]) hanc:

$$d^3v + Adtddv + Bdt^2dv + Cvdt^3 = 0$$

tractassem, ubi quidem A, B, C et dt sunt constantes, esse perductum. Posito scilicet $v = e^{\int xdt}$ obtinueram

$$ddx + 3xdtdx + Adtdx + (x^3 + Axx + Bx + C)dt^2 = 0$$

seu rationem elementi constantis dt exuendo:

$$d\cdot\frac{dx}{dt} + (3x + A)dx + (x^3 + Axx + Bx + C)dt = 0.$$

Nunc posueram $\frac{dx}{dt} = y$ hincque $dt = \frac{dx}{y}$, ex quo nata erat haec aequatio

$$dy + (3x + A)dx + (x^3 + Axx + Bx + C)\frac{dx}{y} = 0$$

sive

$$ydy + (3x + A)ydx + (x^3 + Axx + Bx + C)dx = 0,$$

quae utique in forma proposita comprehenditur.

5. Cum igitur aequatio differentialis tertii ordinis, unde hanc derivavimus, sit integrabilis, ac posito

$$x^3 + Axx + Bx + C = (x - \alpha)(x - \beta)(x - \gamma)$$

eius integrale completum sit

1) Vide Commentationem 188, *Leonhardi Euleri Opera omnia*, series I, vol. 22, p. 185. H. D.

$$v = Fe^{\alpha t} + Ge^{\beta t} + He^{\gamma t},$$

hinc sequitur, etiam nostrae aequationis

$$ydy + (3x + A)ydx + (x^3 + Axx + Bx + C)dx = 0$$

integrale in genere exhiberi posse; quod quidem adeo algebraice ita expressum inde elicui

$$(y + (x - \alpha)(x - \beta))^{\beta-\alpha}(y + (x - \beta)(x - \gamma))^{\gamma-\beta}(y + (x - \gamma)(x - \alpha))^{\alpha-\gamma} = E,$$

ubi notandum est, esse

$$A = -\alpha - \beta - \gamma, \quad B = \alpha\beta + \alpha\gamma + \beta\gamma \quad \text{et} \quad C = -\alpha\beta\gamma.$$

6. Cum enim ob $v = e^{\int x dt}$ sit $x = \frac{dv}{vdt}$, fiet

$$x = \frac{\alpha Fe^{\alpha t} + \beta Ge^{\beta t} + \gamma He^{\gamma t}}{v} \quad \text{et ob} \quad y = \frac{dx}{dt}$$

$$y = \frac{(\beta - \alpha)^2 FGe^{(\alpha+\beta)t} + (\gamma - \alpha)^2 FHe^{(\alpha+\gamma)t} + (\gamma - \beta)^2 GHe^{(\beta+\gamma)t}}{vv}.$$

Inde porro colligitur:

$$x - \alpha = \frac{(\beta - \alpha)Ge^{\beta t} + (\gamma - \alpha)He^{\gamma t}}{v} \quad \text{et} \quad x - \beta = \frac{(\alpha - \beta)Fe^{\alpha t} + (\gamma - \beta)He^{\gamma t}}{v},$$

sicque conficitur

$$y + (x - \alpha)(x - \beta) = \frac{(\gamma - \alpha)(\gamma - \beta)He^{\gamma t}}{v}$$

similique modo

$$y + (x - \beta)(x - \gamma) = \frac{(\alpha - \beta)(\alpha - \gamma)Fe^{\alpha t}}{v}$$

et

$$y + (x - \gamma)(x - \alpha) = \frac{(\beta - \gamma)(\beta - \alpha)Ge^{\beta t}}{v},$$

unde veritas integralis exhibiti fit manifesta et, quia id involvit quantitatem constantem E per arbitrarias F, G, H definitam, pro completo erit habendum.

7. Haec consideratio viam nobis aperit, a priori ad aequationes formae propositae perveniendi. Sumtis enim tribus functionibus ipsius x, quae sint P, Q, R, statuatur aequatio integralis

$$(y+P)^\lambda (y+Q)^\mu (y+R)^\nu = \text{Const.},$$

unde haec nascitur aequatio differentialis:

$$\frac{\lambda dy + \lambda dP}{y+P} + \frac{\mu dy + \mu dQ}{y+Q} + \frac{\nu dy + \nu dR}{y+R} = 0,$$

quae a fractionibus liberata hanc induit formam:

$$\begin{gathered}(\lambda+\mu+\nu)yydy + ydy(\lambda(Q+R)+\mu(P+R)+\nu(P+Q)) + dy(\lambda QR+\mu PR+\nu PQ)\\ + yy(\lambda dP+\mu dQ+\nu dR) + y(\lambda(Q+R)dP+\mu(P+R)dQ+\nu(P+Q)dR)\\ + \lambda QRdP+\mu PRdQ+\nu PQdR = 0,\end{gathered}$$

ex qua forma proposita resultat statuendo:

1°. $\lambda+\mu+\nu=0$,
2°. $\lambda QR+\mu PR+\nu PQ=0$,
3°. $\lambda dP+\mu dQ+\nu dR=0$ seu $\lambda P+\mu Q+\nu R=\text{Const.}$

8. Si hic ponamus

$$\lambda P+\mu Q+\nu R=a \quad \text{et} \quad P+Q+R=S,$$

ut singulas litteras P, Q, R hinc definire valeamus, ratione habita primae conditionis, qua esse oportet

$$\lambda+\mu+\nu=0,$$

reperiemus hos valores:

$$\begin{aligned}P &= \frac{3\lambda\mu\nu S+(\lambda\lambda+2\mu\nu)a+(\mu-\nu)\sqrt{((\lambda\lambda-\mu\nu)aa-3\lambda\mu\nu aS)}}{9\lambda\mu\nu}\\ Q &= \frac{3\lambda\mu\nu S+(\mu\mu+2\lambda\nu)a+(\nu-\lambda)\sqrt{((\mu\mu-\lambda\nu)aa-3\lambda\mu\nu aS)}}{9\lambda\mu\nu}\\ R &= \frac{3\lambda\mu\nu S+(\nu\nu+2\lambda\mu)a+(\lambda-\mu)\sqrt{((\nu\nu-\lambda\mu)aa-3\lambda\mu\nu aS)}}{9\lambda\mu\nu},\end{aligned}$$

ubi signa radicalia ob $\lambda\lambda-\mu\nu=\mu\mu-\lambda\nu=\nu\nu-\lambda\mu$ inter se conveniunt.

9. Irrationalitate harum formularum sublata ad eandem aequationem pervenitur, cuius integrale supra exhibui (§ 5), unde hanc evolutionem ulterius

non prosequor. Interim tamen maximi momenti esse arbitror, observasse aequationem differentialem generalem § 7 expositam per se reddi integrabilem, si ea dividatur per

$$(y + P)(y + Q)(y + R),$$

quod, si ad aequationem superiorem

$$ydy + (3x + A)ydx + (x^3 + Axx + Bx + C)dx = 0$$

attendamus, reperiemus, eam per se integrabilem reddi, si dividatur per hanc formam:

$$y^3 + yy(3xx + 2Ax + B) + y(3x + A)(x^3 + Axx + Bx + C) + (x^3 + Axx + Bx + C)^2,$$

etiamsi hinc minus pateat, integrale adeo algebraice exhiberi posse. Quae observatio me deducit ad methodum illam generalem iam dudum a me expositam, qua ostendi, omnium aequationum differentialium integrationes commodissime per multiplicatores absolvi posse.

10. Cum igitur hic multiplicator seu divisor ita comparatus esse debeat, ut formula per se fiat integrabilis, utique necesse est, ut criteria huiusmodi formularum perspecta habeamus, quae integrabilitatem certo indicent, etiamsi forte ipsa integratio difficulter ac nonnisi per quadraturas satis complicatas confici queat. Omnium autem formularum integrabilium, cuiuscunque gradus differentialia implicent, hanc esse proprietatem nuper demonstravi[1]), ut positis

$$dy = pdx, \quad dp = qdx, \quad dq = rdx \text{ etc.},$$

quo pacto eae semper ad talem formam Vdx reducuntur, in qua littera V utcunque quantitates x, y, p, q, r etc. implicabit, tam futurum sit

$$\left(\frac{dV}{dy}\right) - \frac{1}{dx}d\cdot\left(\frac{dV}{dp}\right) + \frac{1}{dx^2}dd\cdot\left(\frac{dV}{dq}\right) - \frac{1}{dx^3}d^3\cdot\left(\frac{dV}{dr}\right) + \text{etc.} = 0,$$

ac vicissim quoties haec conditio locum habeat, toties quoque formulam Vdx esse integrabilem.

1) Vide L. EULERI Commentationes 297, 420 *Analytica explicatio methodi maximorum et minimorum*; *Methodus nova et facilis calculum variationum tractandi*, § 15. Novi Comment. acad. sc. Petrop. **10**, 1766, p. 94; **16**, 1772, p. 35. Vide quoque *Institutionum calculi integralis* vol. III. Appendix de calculo variationum § 92. *LEONHARDI EULERI Opera omnia*, series I, vol. **24** et **13**. H. D.

11. Hanc igitur quaestionem nunc evolvendam suscipio.

Investigare eiusmodi functionem Z binarum variabilium x et y, per quam aequatio nostra

$$ydy + Mydx + Ndx = 0$$

divisa fiat integrabilis.

Hoc ergo casu erit

$$V = \frac{yp + My + N}{Z},$$

unde colligitur

$$\left(\frac{dV}{dy}\right) = \frac{p}{Z} - \frac{yp}{ZZ}\left(\frac{dZ}{dy}\right) + \frac{M}{Z} - \frac{(My+N)}{ZZ}\left(\frac{dZ}{dy}\right),$$

tum vero ob $\left(\frac{dV}{dp}\right) = \frac{y}{Z}$ porro differentiando reperitur

$$\frac{1}{dx}d\cdot\left(\frac{dV}{dp}\right) = \frac{p}{Z} - \frac{yp}{ZZ}\left(\frac{dZ}{dy}\right) - \frac{y}{ZZ}\left(\frac{dZ}{dx}\right) \quad \text{ob} \quad \frac{dZ}{dx} = \left(\frac{dZ}{dx}\right) + p\left(\frac{dZ}{dy}\right).$$

Cum itaque fieri oporteat

$$\left(\frac{dV}{dy}\right) - \frac{1}{dx}d\cdot\left(\frac{dV}{dp}\right) = 0,$$

habebimus hanc aequationem pro definienda functione Z:

$$\frac{M}{Z} - \frac{(My+N)}{ZZ}\left(\frac{dZ}{dy}\right) + \frac{y}{ZZ}\left(\frac{dZ}{dx}\right) = 0$$

seu

$$MZ - My\left(\frac{dZ}{dy}\right) - N\left(\frac{dZ}{dy}\right) + y\left(\frac{dZ}{dx}\right) = 0.$$

12. Si loco divisoris Z sumatur potestas quaecunque Z^n, ut integrabilis reddi debeat haec forma[1]):

$$\frac{ydy + Mydx + Ndx}{Z^n},$$

functionem Z ex hac aequatione definiri oportebit:

$$MZ - n(My + N)\left(\frac{dZ}{dy}\right) + ny\left(\frac{dZ}{dx}\right) = 0,$$

1) Cf. Commentationem 269 indicis *Enestroemiani* § 93—98. *De integratione aequationum.* Novi Comment. acad. sc. Petrop. 8, 1763, p. 60 = Op. omnia I 22, p. 390. Vide quoque *Institutionum calculi integralis* vol. I, § 493—527. LEONHARDI EULERI *Opera omnia*, series I, vol. 11. H. D.

unde vicissim investigatio ita institui poterit, ut sumta pro lubitu forma functionis Z, inde indoles quantitatum M et N, quae per solam variabilem x determinantur, quaeratur, ut aequatio proposita hoc modo integrabilis reddi queat. Quamobrem his vestigiis insistens sequentes casus evolvam, ubi quidem litteris P, Q, R etc. functiones solius variabilis x indicari moneo.

CASUS 1

Quo integrabilis reddi debet haec forma:

$$\frac{ydy + Mydx + Ndx}{(y+P)^n}.$$

13. Cum ergo sit $Z = y + P$, erit

$$\left(\frac{dZ}{dy}\right) = 1 \quad \text{et} \quad \left(\frac{dZ}{dx}\right) = \frac{dP}{dx},$$

unde paragraphus praecedens hanc suppeditat aequationem:

$$0 = My + MP - nMy - nN + \frac{nydP}{dx};$$

quoniam igitur M, N et P sunt functiones solius x, seorsim esse debet:

1°. $ndP = (n-1)Mdx$ et 2°. $nNdx = MPdx$.

Quare pro lubitu sumta functione P habebimus

$$Mdx = \frac{n}{n-1}dP \quad \text{et} \quad Ndx = \frac{1}{n-1}PdP,$$

unde discimus hanc aequationem:

$$ydy + \frac{n}{n-1}ydP + \frac{1}{n-1}PdP = 0$$

integrabilem reddi, si dividatur per formam $(y+P)^n$.

14. Haec autem aequatio nullam plane habet difficultatem, quoniam est homogenea, atque adeo per hanc formam:

$$(n-1)yy + nyP + PP$$

divisa integrabilis evadat, ex quo divisore, quia constat factoribus

$$(y+P)((n-1)y+P),$$

deducitur aequatio

$$\frac{1}{n-2}\cdot\frac{dy+dP}{y+P}-\frac{1}{(n-1)(n-2)}\cdot\frac{(n-1)dy+dP}{(n-1)y+P}=0,$$

cuius integrale manifesto est

$$(y+P)^{n-1}=A((n-1)y+P),$$

quae etiam ex illo divisore concluditur. Tantum observo casu $n=2$, quo haec forma fit incongrua, integrale fore

$$l(y+P)-\frac{y}{y+P}=\text{Const.},$$

quippe cuius differentiatio praebet

$$\frac{ydy+2ydP+PdP}{(y+P)^2}=0.$$

15. Singularis hic se obtulit casus, quo aequatio

$$(y+P)^{n-1}=A((n-1)y+P),$$

quae ob constantem arbitrariam A est indefinita, sumto $n=2$ hac indole penitus privatur. Ut autem tum eius vera forma eliciatur, statuatur more solito $n=2+\omega$, denotante ω fractionem infinite parvam, ut sit

$$(y+P)^{\omega}=1+\omega\, l(y+P);$$

sic illa aequatio hanc induet formam:

$$y+P+\omega(y+P)l(y+P)=A(y+P)+A\omega y,$$

sit nunc $A=1+B\omega$ ac prodibit

$$(y+P)l(y+P)=y+B(y+P),$$

sicque loco constantis A alia arbitraria B est introducta.

16. Ut aequationem inventam elegantiorem reddamus, ponamus $P = x^{n-1}$, ut prodeat

$$y\,dy + nx^{n-2}y\,dx + x^{2n-3}dx = 0,$$

quam ergo integrabilem fieri novimus, si dividatur per $(y + x^{n-1})^n$, et integrale eius completum erit

$$(y + x^{n-1})^{n-1} = A\,((n-1)y + x^{n-1}),$$

dum observetur casu $n = 2$ integrale esse

$$l(y + x^{n-1}) - \frac{y}{y + x^{n-1}} = \text{Const.} = l(y + x) - \frac{y}{y + x}\cdot$$

Hic vero novus casus singularis occurrit $n = 1$, aequationem praebens

$$y\,dy + \frac{y\,dx}{x} + \frac{dx}{x} = 0\,,$$

quae per $y + 1$ divisa integrale dat

$$y + l\frac{x}{y+1} = \text{Const.}$$

Quod ut ex forma illa generali eruatur, posito $n = 1 + \omega$ erit

$$(y + 1 + \omega\, lx)^{\omega} = 1 + \omega\, l(y + 1) = A(\omega\, y + 1 + \omega\, lx),$$

sit ergo $A = 1 + B\omega$ prodibitque

$$l(y + 1) = y + lx + B \quad \text{seu} \quad y + l\frac{x}{y+1} = \text{Const.}$$

ut ante.

CASUS 2

Quo integrabilis reddenda est haec forma:

$$\frac{y\,dy + My\,dx + N\,dx}{(yy + Py + Q)^n}\cdot$$

17. Quia hic est

$$Z = yy + Py + Q,$$

pro § 12 habebimus

$$\left(\frac{dZ}{dy}\right) = 2y + P \quad \text{et} \quad \left(\frac{dZ}{dx}\right) = \frac{ydP}{dx} + \frac{dQ}{dx},$$

hincque istam aequationem resolvendam:

$$\begin{array}{rlll} 0 = & Myy + & MPy + & MQ \\ & -2nM & -nMP & -nNP \\ & +\dfrac{ndP}{dx} & -2nN & \\ & & +\dfrac{ndQ}{dx}, & \end{array}$$

unde resultant hae tres:

$$\begin{aligned} ndP &= (2n-1)\,M\,dx \quad \text{seu} \quad M\,dx = \frac{ndP}{2n-1} \\ ndQ &= (n-1)\,MPdx + 2nNdx \\ nNP &= MQ\,, \quad \text{seu } Ndx = \frac{MQdx}{nP} = \frac{QdP}{(2n-1)P}\,. \end{aligned}$$

Ergo

$$ndQ = \frac{n(n-1)PdP}{2n-1} + \frac{2nQdP}{(2n-1)P}$$

seu

$$dQ - \frac{2QdP}{(2n-1)P} = \frac{n-1}{2n-1}PdP\,,$$

quae aequatio per $P^{\frac{-2}{2n-1}}$ multiplicata et integrata dat

$$P^{\frac{-2}{2n-1}}Q = \tfrac{1}{4}P^{\frac{4n-4}{2n-1}} + A\,,$$

hincque fit

$$Q = \tfrac{1}{4}PP + AP^{\frac{2}{2n-1}}$$

et per P resolutio erit

$$M\,dx = \frac{ndP}{2n-1} \quad \text{et} \quad N\,dx = \frac{dP}{2n-1}\left(\tfrac{1}{4}P + AP^{\frac{3-2n}{2n-1}}\right).$$

18. Sit $P = 2x^{2n-1}$, ut fiat

$$Q = x^{4n-2} + Bxx,\ Mdx = 2nx^{2n-2}dx \quad \text{et} \quad Ndx = x^{4n-3}dx + Bxdx\,,$$

atque hinc intelligimus hanc aequationem

$$ydy + 2nx^{2n-2}ydx + x^{4n-3}dx + Bxdx = 0$$

integrabilem reddi, si dividatur per

$$(yy + 2x^{2n-1}y + x^{4n-2} + Bxx)^n.$$

Evidens est hanc aequationem ad simpliciorem formam perduci ponendo $y = z - x^{2n-1}$, tum enim prodit

$$zdz + Bxdx + x^{2n-2}(zdx - xdz) = 0,$$

quae per $(zz + Bxx)^n$ divisa utique fit integrabilis integrali existente

$$\frac{-1}{2(n-1)(zz + Bxx)^{n-1}} + \int \frac{x^{2n-2}(zdx - xdz)}{(zz + Bxx)^n} = 0,$$

cuius posterius membrum posito $z = vx$ abit in

$$-\int \frac{dv}{(vv + B)^n},$$

ita ut nulla supersit difficultas.

CASUS 3

Quo integrabilis reddenda est haec forma:

$$\frac{ydy + Mydx + Ndx}{(y^3 + Py^2 + Qy + R)^n}.$$

19. Ob $Z = y^3 + Py^2 + Qy + R$ erit

$$\left(\frac{dZ}{dy}\right) = 3yy + 2Py + Q \quad \text{et} \quad \left(\frac{dZ}{dx}\right) = \frac{y^2dP + ydQ + dR}{dx},$$

ex quo sequenti aequationi est satisfaciendum

$$\begin{array}{rllll} 0 = & My^3 + & MPy^2 + & MQy + & MR \\ & -3nM & -2nMP & -nMQ & -nNQ \\ & +\dfrac{ndP}{dx} & -3nN & -2nNP & \\ & & +\dfrac{ndQ}{dx} & +\dfrac{ndR}{dx}, & \end{array}$$

quae suppeditat has quatuor determinationes:

$$\begin{aligned}
&1^0.\quad ndP &&= (3n-1)Mdx\\
&2^0.\quad ndQ &&= (2n-1)MPdx + 3nNdx\\
&3^0.\quad ndR &&= \ (n-1)MQdx + 2nNPdx\\
&4^0.\quad nNQdx &&= MRdx.
\end{aligned}$$

20. Ex ultima colligimus $M : N = nQ : R$, unde ex prioribus litteras M et N elidendo obtinemus:

$$\begin{aligned}
(3n-1)QdQ &= (2n-1)PQdP + 3RdP \quad \text{et}\\
(3n-1)QdR &= \ (n-1)QQdP + 2PRdP,
\end{aligned}$$

hincque

$$(2n-1)PQdR + 3RdR = (n-1)QQdQ + 2PRdQ.$$

Unde si quantitates Q et R per P definire licuerit, tum erit

$$Mdx = \frac{n}{3n-1}dP \quad \text{et} \quad Ndx = \frac{RdP}{(3n-1)Q}.$$

Primum autem observo illis aequationibus satisfieri posse ponendo

$$Q = \alpha P^2 \quad \text{et} \quad R = \beta P^3,$$

hosque coefficientes α et β duplicem determinationem sortiri, scilicet

$$\begin{aligned}
&\text{vel } \alpha = \frac{1}{3} && \text{et } \beta = \frac{1}{27}\\
&\text{vel } \alpha = \frac{2n-1}{(3n-1)^2} && \text{et } \beta = -\frac{(n-1)(2n-1)^2}{(3n-1)^3},
\end{aligned}$$

quo quidem casu aequatio nostra fit homogenea.

21. Consideremus primo casum quo $n = 1$, quippe quem iam supra aliunde elicuimus; eruntque nostrae aequationes:

$$1^0.\ 2QdQ = PQdP + 3RdP \quad \text{et} \quad 2^0.\ QdR = PRdP$$

hincque

$$3^0.\ PQdR + 3RdR = 2PRdQ.$$

Iam ex 2°. fit

$$Q = \frac{PRdP}{dR},$$

sumtoque dR constante

$$dQ = \frac{PRddP}{dR} + PdP + \frac{RdP^2}{dR},$$

unde prima abit in hanc:

$$\frac{2PPRRdPddP}{dR^2} + \frac{2PPRdP^2}{dR} + \frac{2PRRdP^3}{dR^2} - \frac{PPRdP^2}{dR} = 3RdP$$

seu

$$\frac{2PPRRddP + 2PRRdP^2}{dR} + PPRdP = 3RdR;$$

dividatur per $PR\sqrt{R}$, ut prodeat

$$\frac{2PddP\sqrt{R}}{dR} + \frac{2dP^2\sqrt{R}}{dR} + \frac{PdP}{\sqrt{R}} = \frac{3dR}{P\sqrt{R}},$$

cuius prius membrum integratum dat

$$\frac{2PdP\sqrt{R}}{dR} = 3\int\frac{dR}{P\sqrt{R}} \quad \text{hincque} \quad dP = \frac{3}{2}\cdot\frac{dR}{P\sqrt{R}}\int\frac{dR}{P\sqrt{R}},$$

quae quidem forma parum lucri attulisse videtur.

22. Ponamus autem

$$\int\frac{dR}{P\sqrt{R}} = u, \quad \text{ut sit} \quad P = \frac{dR}{du\sqrt{R}},$$

ac postrema aequatio $dP = \frac{3}{2}udu$ dabit

$$P = \tfrac{3}{4}uu + A,$$

unde fit

$$\frac{dR}{\sqrt{R}} = \tfrac{3}{4}uudu + Adu,$$

hincque integrando

$$2\sqrt{R} = \tfrac{1}{4}u^3 + Au + 2B \quad \text{et} \quad R = (\tfrac{1}{8}u^3 + \tfrac{1}{2}Au + B)^2$$

et ob

$$Q = \frac{PRdP}{dR} = \frac{PdP\sqrt{R}}{Pdu} = \tfrac{3}{2}u\sqrt{R}$$

erit

$$Q = \tfrac{3}{2}u\left(\tfrac{1}{8}u^3 + \tfrac{1}{2}Au + B\right)$$

ac denique

$$Mdx = \tfrac{1}{2}dP = \tfrac{3}{4}udu$$

et

$$Ndx = \tfrac{1}{2}dP\cdot\frac{R}{Q} = \tfrac{1}{2}\frac{dR}{P} = \tfrac{1}{2}du\sqrt{R} = \tfrac{1}{2}du\left(\tfrac{1}{8}u^3 + \tfrac{1}{2}Au + B\right);$$

statuamus nunc $u = 2x + 2f$, ut fiat

$$P = 3xx + 6fx + 3ff + A;$$

$$Q = 3(x+f)\left\{\begin{array}{l} x^3 + 3fxx + 3ffx + f^3 \\ \qquad\qquad\quad + Ax \;\; + Af \\ \qquad\qquad\qquad\quad\;\; + B \end{array}\right\}$$

$$\sqrt{R} = \left\{\begin{array}{l} x^3 + 3fxx + 3ffx + f^3 \\ \qquad\qquad\quad + Ax \;\; + Af \\ \qquad\qquad\qquad\quad\;\; + B \end{array}\right\}$$

$$Mdx = 3dx(x+f)$$

et

$$Ndx = dx\left\{\begin{array}{l} x^3 + 3fxx + 3ffx + f^3 \\ \qquad\qquad\quad + Ax \;\; + Af \\ \qquad\qquad\qquad\quad\;\; + B \end{array}\right\}.$$

23. Ponamus porro:

$$3f = a, \quad 3ff + A = b \quad \text{et} \quad f^3 + Af + B = c,$$

atque obtinebimus hanc aequationem:

$$ydy + (3x + a)ydx + (x^3 + axx + bx + c)dx = 0,$$

quae ergo integrabilis reddetur, si dividatur per

$$y^3 + (3xx + 2ax + b)yy + (3x + a)(x^3 + axx + bx + c)y + (x^3 + axx + bx + c)^2,$$

sicque divisoris forma hoc casu assumta ad eam ipsam integrationem nos deduxit, quam iam supra § 9 sumus adepti. Circa hunc ergo divisorem annotasse iuvabit, quod supra iam vidimus, si ponatur

$$x^3 + axx + bx + c = (x + \alpha)(x + \beta)(x + \gamma),$$

ut sit

$$a = \alpha + \beta + \gamma, \quad b = \alpha\beta + \alpha\gamma + \beta\gamma \quad \text{et} \quad c = \alpha\beta\gamma,$$

divisorem nostrum sic in ternos factores resolutum exhiberi posse

$$(y + (x + \alpha)(x + \beta))(y + (x + \alpha)(x + \gamma))(y + (x + \beta)(x + \gamma))$$

et aequationis nostrae integrale completum fore

$$(y + (x + \alpha)(x + \beta))^{\alpha-\beta}(y + (x + \beta)(x + \gamma))^{\beta-\gamma}(y + (x + \gamma)(x + \alpha))^{\gamma-\alpha} = \text{Const.}$$

24. Circa aequationem differentio-differentialem § 21, quae per R divisa est

$$\frac{2PPRddP + 2PRdP^2 + PPdPdR}{dR} = 3dR,$$

observo, eam multo facilius integrari posse, si modo per $\frac{dP}{dR}$ multiplicetur, ut habeatur

$$\frac{2PPRdPddP + 2PRdP^3 + PPdP^2dR}{dR^2} = 3dP,$$

cuius integrale statim est

$$\frac{PPRdP^2}{dR^2} = 3P + A, \text{ hincque } \frac{dR}{\sqrt{R}} = \frac{PdP}{\sqrt{(3P + A)}},$$

cuius integrale denuo est

$$2\sqrt{R} = 2\left(\frac{P}{9} - \frac{2A}{27}\right)\sqrt{(3P + A)} + B.$$

25. Pro positione $n = 1$ calculum in genere expedire licuit, pro aliis autem valoribus ipsius n negotium minus succedit, excepto unico casu $n = \frac{1}{3}$, quo aequatio prima § 20 statim dat $dP = 0$ ideoque $P = a$; unde inter Q et R haec prodit aequatio

$$-\frac{1}{3}aQdR + 3RdR = -\frac{2}{3}QQdQ + 2aRdQ$$

seu

$$9RdR + 2QQdQ = 6aRdQ + aQdR,$$

quam autem evolvere non licet, nisi sumatur $a = 0$, tum autem oritur

$$\frac{9}{2}RR + \frac{2}{3}Q^3 = \text{Const.} \text{ seu } R = \sqrt{(A - \frac{4}{27}Q^3)},$$

deinde
$$N\,dx = \tfrac{1}{3}dQ \quad \text{et} \quad M\,dx = \tfrac{1}{9}dQ \cdot \frac{Q}{R}.$$
Statuamus
$$Q = 3x\,, \quad \text{ut fiat} \quad R = \sqrt{(A - 4x^3)}\,, \quad N\,dx = dx \quad \text{et} \quad M\,dx = \frac{x\,dx}{\sqrt{(A - 4x^3)}}\,,$$
unde deducimus hanc aequationem differentialem:
$$y\,dy + \frac{xy\,dx}{\sqrt{(A - 4x^3)}} + dx = 0\,,$$
quam nunc novimus integrabilem reddi, si dividatur per
$$\sqrt[3]{(y^3 + 3xy + \sqrt{(A - 4x^3)})}.$$
Hinc autem ipsum integrale neutiquam explicite exhiberi potest, quin etiam aequatio ista ita est comparata, ut nulla alia via ad constructionem perduci queat.

CASUS 4

Quo integrabilis reddenda est haec forma:
$$\frac{y\,dy + My\,dx + N\,dx}{(y^4 + Py^3 + Qy^2 + Ry + S)^n}.$$

26. Pro hoc casu methodus nostra sequentes suppeditat aequationes:
$$\begin{aligned}
&1^0.\ n\,dP = (4n - 1)M\,dx\\
&2^0.\ n\,dQ = (3n - 1)MP\,dx + 4nN\,dx\\
&3^0.\ n\,dR = (2n - 1)MQ\,dx + 3nNP\,dx\\
&4^0.\ n\,dS = (n - 1)MR\,dx + 2nNQ\,dx\\
&5^0.\ MS = nNR \quad \text{seu} \quad M : N = nR : S,
\end{aligned}$$
unde derivantur istae:
$$\begin{aligned}
(4n - 1)R\,dQ &= (3n - 1)PR\,dP + 4S\,dP\\
(4n - 1)R\,dR &= (2n - 1)QR\,dP + 3PS\,dP\\
(4n - 1)R\,dS &= (n - 1)RR\,dP + 2QS\,dP,
\end{aligned}$$
ex binis postremis elidendo Q oritur
$$2(4n-1)RS\,dR - (2n-1)(4n-1)RR\,dS = 6PSS\,dP - (n-1)(2n-1)R^3\,dP.$$

27. Quoniam hic in genere vix quicquam concludere licet praeter casum homogeneitatis, quo fieri potest

$$Q = \alpha P^2,\ R = \beta P^3 \quad \text{et} \quad S = \gamma P^4,$$

consideremus casum $n = 1$, ut divisor sit

$$y^4 + Py^3 + Qy^2 + Ry + S.$$

Habebimus ergo

$$M\,dx = \tfrac{1}{3}dP,\ N\,dx = \frac{SdP}{3R} = \frac{dS}{2Q}$$

et

$$\begin{aligned} 1^0.\ 3RdQ &= 2PRdP + 4SdP \\ 2^0.\ 3RdR &= QRdP + 3PSdP \\ 3^0.\ 3RdS &= 2QSdP, \end{aligned}$$

unde eliminato Q ex duabus postremis fit

$$2RSdR - RRdS = 2PSSdP,$$

quae per SS divisa commode integrationem admittit praebetque

$$\frac{RR}{S} = PP + A \quad \text{seu} \quad S = \frac{RR}{PP + A},$$

quare ex secunda elicitur

$$Q = \frac{3\,dR}{dP} - \frac{3PS}{R} = \frac{3\,dR}{dP} - \frac{3PR}{PP + A}.$$

28. Prima vero dat

$$dQ = \frac{2}{3}PdP + \frac{4SdP}{3R} = \frac{2}{3}PdP + \frac{4\,RdP}{3(PP + A)},$$

inde vero sumendo dP constans reperitur

$$dQ = \frac{3\,ddR}{dP} - \frac{3\,PdR}{PP + A} + \frac{3\,PPRdP - 3ARdP}{(PP + A)^2},$$

sicque oritur haec aequatio differentialis secundi gradus:

$$\frac{ddR}{dP}-\frac{PdR}{PP+A}+\frac{5PPRdP-13ARdP}{9(PP+A)^2}-\frac{2}{9}PdP=0,$$

quam dubito in genere resolvi posse.

29. Considerabo ergo casum quo $A=0$, ideoque

$$S=\frac{RR}{PP} \quad \text{et} \quad Q=\frac{3dR}{dP}-\frac{3R}{P},$$

ita ut haec resolvenda sit aequatio:

$$\frac{ddR}{dP}-\frac{dR}{P}+\frac{5RdP}{9PP}-\frac{2}{9}PdP=0.$$

Statuamus ergo $R=\alpha P^3+u$ fietque

$$\frac{ddu}{dP}-\frac{du}{P}+\frac{5udP}{9PP}+6\alpha PdP-3\alpha PdP+\frac{5}{9}\alpha PdP-\frac{2}{9}PdP=0$$

et sumto $\alpha=\frac{1}{16}$ erit

$$\frac{ddu}{dP}-\frac{du}{P}+\frac{5udP}{9PP}=0,$$

pro qua porro $u=P^\lambda$ et ex aequalitate

$$\lambda\lambda-2\lambda+\frac{5}{9}=0$$

colligo

$$\lambda=1\pm\frac{2}{3},$$

hincque integrale completum

$$u=\alpha P^{\frac{5}{3}}+\beta P^{\frac{1}{3}}.$$

Quocirca habebimus:

$$R=\frac{1}{16}P^3+\alpha P^{\frac{5}{3}}+\beta P^{\frac{1}{3}},$$

$$Q=\frac{3}{8}P^2+2\alpha P^{\frac{2}{3}}-2\beta P^{-\frac{2}{3}},$$

$$S=\left(\frac{1}{16}P^2+\alpha P^{\frac{2}{3}}+\beta P^{-\frac{2}{3}}\right)^2;$$

ac tandem

$$M\,dx = \tfrac{1}{3}\,dP,$$

et

$$N\,dx = \frac{dP}{3P}\left(\frac{1}{16}P^2 + \alpha P^{\frac{2}{3}} + \beta P^{-\frac{2}{3}}\right).$$

30. Statuamus nunc $P = t^3$, ut sublata irrationalitate fiat:

$$M\,dx = ttdt \quad \text{et} \quad N\,dx = \frac{dt}{t}\left(\frac{1}{16}t^6 + \alpha tt + \frac{\beta}{t^2}\right)$$

et aequatio nostra huius sit formae:

$$ydy + ytt\,dt + \frac{dt}{t^3}\left(\frac{1}{16}t^8 + \alpha t^4 + \beta\right) = 0,$$

quam iam novimus integrabilem reddi, si dividatur per

$$y^4 + t^3y^3 + \left(\frac{3}{8}t^6 + 2\alpha tt - \frac{2\beta}{tt}\right)y^2 + \left(\frac{1}{16}t^9 + \alpha t^5 + \beta t\right)y + \left(\frac{1}{16}t^6 + \alpha tt + \frac{\beta}{tt}\right)^2.$$

Hic divisor duobus constat factoribus:

$$\left(yy + \left(\frac{1}{2}t^3 + \frac{2}{t}\sqrt{\beta}\right)y + \frac{1}{16}t^6 + \alpha tt + \frac{\beta}{tt}\right)\left(yy + \left(\frac{1}{2}t^3 - \frac{2}{t}\sqrt{\beta}\right)y + \frac{1}{16}t^6 + \alpha tt + \frac{\beta}{tt}\right).$$

Si divisores magis complicatos adhibere vellemus, vix quicquam ad usum inde concludere liceret.

CONSIDERATIO AEQUATIONIS DIFFERENTIO-DIFFERENTIALIS

$$(a+bx)\,ddz+(c+ex)\frac{dx\cdot dz}{x}+(f+gx)\frac{z\,dx^2}{xx}=0$$

Commentatio 431 indicis ENESTROEMIANI
Novi Commentarii academiae scientiarum Petropolitanae **17** (1772), 1773, p. 125—154
Summarium ibidem p. 12

SUMMARIUM

Cum ea, quae hac dissertatione continentur, sint eiusmodi, ut meris absolvantur calculis algebraicis, commode fieri non potest, ut hic eorum tradatur epitome, unde mathematum peritos ad ipsam dissertationem necesse est ut ablegemus.

1. Primo haec aequatio ad formam differentialem simplicem revocari potest ponendo $lz=\int v\,dx$, ut sit

$$\frac{dz}{z}=v\,dx \quad \text{et} \quad \frac{ddz}{z}-\frac{dz^2}{zz}=dx\,dv,$$

ideoque

$$\frac{ddz}{z}=dx\,dv+vv\,dx^2.$$

Divisa enim illa aequatione per $z\,dx$ hinc orietur

$$(a+bx)\,dv+(a+bx)\,vv\,dx+(c+ex)\frac{v\,dx}{x}+(f+gx)\frac{dx}{xx}=0,$$

cuius integratio si pateret, foret pro proposita $lz=\int v\,dx$.

2. Hinc duplici modo terminus simplici quantitate v affectus elidi potest. Pro altero ponamus $v = u + X$ denotante X functionem ipsius x mox determinandam, et facta substitutione fiet

$$\left.\begin{aligned}(a+bx)du + (a+bx)uudx + (c+ex)\frac{udx}{x} &+ (f+gx)\frac{dx}{xx}\\ + 2(a+bx)uXdx &+ (a+bx)dX\\ &+ (a+bx)XXdx\\ &+ (c+ex)\frac{Xdx}{x}\end{aligned}\right\} = 0.$$

Iam statuatur

$$X = \frac{-(c+ex)}{2x(a+bx)},$$

ut sit

$$dX = \frac{(ac+2bcx+bexx)dx}{2xx(a+bx)^2},$$

prodibitque

$$\left.\begin{aligned}(a+bx)du + (a+bx)uudx + (f+gx)\frac{dx}{xx} &+ \frac{(ac+2bcx+bexx)dx}{2xx(a+bx)}\\ &- \frac{(cc+2cex+eexx)dx}{4xx(a+bx)}\end{aligned}\right\} = 0$$

seu hoc modo:

$$du + uudx + \frac{4(a+bx)(f+gx)+2(ac+2bcx+bexx)-(c+ex)^2}{4xx(a+bx)^2}dx = 0.$$

3. Per alteram methodum cum priore coniunctam ponatur $v = Pu + X$, existentibus P et X functionibus ipsius x, et substitutione facta obtinebitur

$$\left.\begin{aligned}(a+bx)Pdu + (a+bx)PPuudx + (c+ex)\frac{Pudx}{x} &+ \frac{(f+gx)dx}{xx}\\ + (a+bx)udP &+ (a+bx)dX\\ + 2(a+bx)PXudx &+ (a+bx)XXdx\\ &+ (c+ex)\frac{Xdx}{x}\end{aligned}\right\} = 0,$$

unde fieri debet

$$\frac{(c+ex)Pdx}{x} + (a+bx)dP + 2(a+bx)PXdx = 0.$$

Introduxi autem hic binas functiones P et X, quo investigatio latius pateat; vulgo enim hac altera methodo utentes ponere solemus $v = Pu$, ut sit $X = 0$, quo casu erit

$$\frac{dP}{P} + \frac{(c+ex)\,dx}{x(a+bx)} = 0 \quad \text{seu} \quad \frac{dP}{P} + \frac{c}{a}\cdot\frac{dx}{x} + \frac{ae-bc}{a}\cdot\frac{dx}{a+bx} = 0,$$

unde integrando colligitur:

$$Px^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}} = C \quad \text{et} \quad P = Cx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}$$

et aequatio nostra differentialis primi gradus erit

$$Cx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}+1}du + CCx^{\frac{-2c}{a}}(a+bx)^{\frac{2c}{a}-\frac{2e}{b}+1}uudx + \frac{(f+gx)\,dx}{xx} = 0$$

sive hoc modo

$$Cdu + CCx^{\frac{-c}{a}}(a+bx)^{\frac{c}{a}-\frac{e}{b}}uudx + x^{\frac{c}{a}}(a+bx)^{\frac{e}{b}-\frac{c}{a}-1}(f+gx)\frac{dx}{xx} = 0.$$

4. Sin autem in genere ponatur $v = Pu + X$ statuaturque

$$\frac{dP}{P} + \frac{(c+ex)\,dx}{x(a+bx)} + 2X\,dx = 0,$$

aequatio nostra differentialis hanc induit formam:

$$Pdu + PPuudx + dX + XXdx + \frac{(c+ex)\,Xdx}{x(a+bx)} + \frac{(f+gx)\,dx}{xx(a+bx)} = 0,$$

in qua vel P vel X pro lubitu accipi potest, unde altera definietur. Veluti si capiatur $P = \alpha x^n$, fiet

$$X = \frac{-n}{2x} - \frac{c+ex}{2x(a+bx)} = \frac{-na-c-(nb+e)x}{2x(a+bx)}.$$

Ex his formis casus, quibus aequatio fit integrabilis, elicere licet, quos autem facilius ex ipsa aequatione proposita cognoscere poterimus.

5. Quodsi quaerere velimus casus, quibus aequatio proposita integrationem admittit, in quo quidem omnis opera collocanda videtur, quamdiu

integrationem in genere instituere non licet, primum quidem statim se offert forma

$$z = Ax^m (a+bx)^n,$$

quae ut satisfaciat, definiri oportet relationem quantitatum constantium a, b, c, e, f, g. Cum igitur sit

$$\frac{dz}{z} = \frac{m\,dx}{x} + \frac{nb\,dx}{a+bx} \quad \text{et} \quad \frac{ddz}{z} - \frac{dz^2}{zz} = \frac{-m\,dx^2}{xx} - \frac{nbb\,dx^2}{(a+bx)^2},$$

erit

$$\frac{ddz}{z} = \frac{m(m-1)dx^2}{xx} + \frac{2mnb\,dx^2}{x(a+bx)} + \frac{n(n-1)bb\,dx^2}{(a+bx)^2},$$

hincque nascitur facta divisione per dx^2 haec aequatio:

$$\left.\begin{aligned} &\frac{m(m-1)(a+bx)}{xx} + \frac{2mnb}{x} \qquad + \frac{n(n-1)bb}{a+bx} \\ &+ \frac{f+gx}{xx} \qquad + \frac{m(c+ex)}{xx} + \frac{nb(c+ex)}{x(a+bx)} \end{aligned}\right\} = 0,$$

quae ut subsistere possit, bini postremi termini collecti

$$\frac{nb((n-1)bx+c+ex)}{x(a+bx)}$$

denominatorem $a+bx$ amittere debent, ex quo fit

$$a:c = b:(n-1)b+e, \quad \text{seu} \quad n-1 = \frac{c}{a} - \frac{e}{b},$$

et

$$n = 1 + \frac{c}{a} - \frac{e}{b},$$

ita ut habeatur haec aequatio:

$$\frac{m(m-1)a+mc+f}{xx} + \frac{m(m-1)b+2mnb+me+g+nbc:a}{x} = 0,$$

unde hae duae nascuntur aequationes loco n valorem inventum scribendo:

$$m(m-1)a+mc+f = 0$$

et

$$m(m+1)b+\frac{(2m+1)bc}{a}-me-\frac{ce}{a}+\frac{bcc}{aa}+g=0.$$

Multiplicetur haec per a et illa per $-b$, fiet summa

$$2mab+mbc-mae+bc-ce+\frac{bcc}{a}+ag-bf=0,$$

hincque

$$m=\frac{abf-aag-abc+ace-bcc}{2aab+abc-aae}$$

et

$$(m-1)a+c=\frac{abf-aag-2aab+aae}{2ab+bc-ae},$$

qui valores in prima substituti dant

$$(bf-ag)^2+f(2abb+3bbc-3abe-bce+aee)+c(2b-e)(ab-ae+bc)+g(2aab-aae+abc-ace+bcc)=0,$$

quae resoluta praebet

$$ag=bf-\tfrac{1}{2}(2ab-ae+bc)+\frac{c}{2a}(ae-bc)\pm(2ab-ae+bc)\sqrt{\left(\frac{(a-c)^2}{4aa}-\frac{f}{a}\right)}$$

sive

$$g=\frac{c(ae-bc)+2abf-(2ab-ae+bc)\ (a\pm\sqrt{((a-c)^2-4af)})}{2aa}.$$

Quare si littera g hunc habeat valorem, aequatio nostra integrale habebit

$$z=Ax^m(a+bx)^n$$

existente

$$m=\frac{a-c\pm\sqrt{((a-c)^2-4af)}}{2a}\quad\text{et}\quad n=1+\frac{c}{a}-\frac{e}{b}.$$

6. Alia via casus integrabiles reperiuntur, si valor ipsius z in seriem convertatur[1]), quae si alicubi abrumpatur, expressionem finitam pro z exhibet. Fingatur ergo

$$z=Ax^n+Bx^{n+1}+Cx^{n+2}+Dx^{n+3}+Ex^{n+4}+\text{etc.}$$

1) Cf. *Institutionum calculi integralis* vol. II, § 929—991, 997—1016, 1035—1036, 1069—1076. Vide quoque Commentationes 95, 284 (indicis *Enestroemiani*). *De aequationibus differentialibus, quae certis tantum casibus integrationem admittunt.* Comment. acad. Petrop. 10, 1747, p. 40. *De resolutione aequationis $dy+ayydx=bx^m dx$.* Novi Comment. acad. sc. Petrop. 9, 1764, p. 155. LEONHARDI EULERI *Opera omnia*, series I, vol. 12, 22. H. D.

et facta substitutione consequemur:

$$\left.\begin{array}{ccccccc}
n(n-1)Aax^{n-2} & +(n+1)nBax^{n-1} & + & (n+2)(n+1)Cax^{n} & + & (n+3)(n+2)Dax^{n+1} & \text{etc.}\\
 & +n(n-1)Ab & + & (n+1)nBb & + & (n+2)(n+1)Cb & \\
+nAc & +\ (n+1)Bc & + & (n+2)Cc & + & (n+3)Dc & \\
 & +\quad nAe & + & (n+1)Be & + & (n+2)Ce & \\
+Af & +\quad Bf & + & Cf & + & Df & \\
 & +\quad Ag & + & Bg & + & Cg &
\end{array}\right\}=0$$

quos singulos terminos ad nihilum reduci oportet. Primo ergo erit

$$n(n-1)a+nc+f=0,$$

hincque

$$n=\frac{a-c\pm\sqrt{((a-c)^2-4af)}}{2a},$$

porro vero

$$B=\frac{-n(n-1)b-ne-g}{(n+1)na+(n+1)c+f}A=\frac{-n(n-1)b-ne-g}{2na+c}A$$

$$C=\frac{-(n+1)nb-(n+1)e-g}{(n+2)(n+1)a+(n+2)c+f}B=\frac{-(n+1)nb-(n+1)e-g}{2((2n+1)a+c)}B$$

$$D=\frac{-(n+2)(n+1)b-(n+2)e-g}{(n+3)(n+2)a+(n+3)c+f}C=\frac{-(n+2)(n+1)b-(n+2)e-g}{3((2n+2)a+c)}C$$

etc.

Haec ergo series alicubi abrumpitur, si sumto pro i numero quocunque integro positivo, quo etiam cyphra referatur, fuerit

$$g=-(n+i)(n+i-1)b-(n+i)e.$$

Cum autem sit

$$n+i=\frac{(2i+1)a-c\pm\sqrt{((a-c)^2-4af)}}{2a}$$

et

$$n+i-1=\frac{(2i-1)a-c\pm\sqrt{((a-c)^2-4af)}}{2a},$$

erit

$$g=\frac{-((2i+1)a-c\pm\sqrt{((a-c)^2-4af)})\ ((2i-1)ab-bc+2ae\pm b\sqrt{((a-c)^2-4af)})}{4aa}$$

et evolvendo

$$g=\frac{2abf+c(ae-bc)-a(2iiab+(2i+1)(ae-bc))\mp(2iab+ae-bc)\sqrt{((a-c)^2-4af)}}{2aa};$$

si ergo esset $i=-1$, quod autem hic sumere non licet, casus praecedens emergeret. Hinc igitur innumerabiles alii casus similes eruuntur.

7. Possumus etiam seriem, in qua exponentes ipsius x decrescant, assumere hoc modo

$$z=Ax^n+Bx^{n-1}+Cx^{n-2}+Dx^{n-3}+Ex^{n-4}+\text{etc.},$$

qua substituta nostra aequatio fit

$$0=\left\{\begin{array}{llllll} & & n(n-1)Aax^{n-2} & +(n-1)(n-2)Bax^{n-3} & +(n-2)(n-3)Cax^{n-4} & +\text{etc.}\\ +n(n-1)Abx^{n-1} & +(n-1)(n-2)Bb & +(n-2)(n-3)Cb & +(n-3)(n-4)Db & & \\ & + & nAc & + & (n-1)Bc & + \quad (n-2)Cc\\ +\quad nAe & + & (n-1)Be & + & (n-2)Ce & + \quad (n-3)De\\ & + & Af & + & Bf & + \quad Cf\\ +\quad Ag & + & Bg & + & Cg & + \quad Dg \end{array}\right.$$

hincque esse debet

$$n(n-1)b+ne+g=0$$

seu

$$n=\frac{b-e\pm\sqrt{((b-e)^2-4bg)}}{2b},$$

vel

$$g=-nnb+nb-ne.$$

Praeterea vero:

$$B=\frac{n(n-1)a+nc+f}{(2n-2)b+e}A,\qquad C=\frac{(n-1)(n-2)a+(n-1)c+f}{2((2n-3)b+e)}B,$$

$$D=\frac{(n-2)(n-3)a+(n-2)c+f}{3((2n-4)b+e)}C,\quad E=\frac{(n-3)(n-4)a+(n-3)c+f}{4((2n-5)b+e)}D$$

etc.

Sit ut ante i numerus integer positivus, cyphra non exclusa, et integrale finitum obtinebitur, quoties fuerit

$$(n-i)(n-i-1)a+(n-i)c+f=0,$$

unde fit

$$n=\frac{(2i+1)a-c+\sqrt{((a-c)^2-4af)}}{2a}\text{ [1]},$$

ut invenimus $g=-n((n-1)b+e)$, ideoque

$$g=\frac{-((2i+1)a-c+\sqrt{((a-c)^2-4af)})((2i-1)ab-bc+2ae+b\sqrt{((a-c)^2-4af)})}{4aa},$$

quae evoluta praebet ut ante

$$g=\frac{2abf+c(ae-bc)-a(2iiab+(2i+1)(ae-bc))-(2iab+ae-bc)\sqrt{((a-c)^2-4af)}}{2aa},$$

ita ut hinc iidem casus ac ante prodeant, atque adeo eadem integralia ordine retrogrado scripta obtineantur.

8. Verum ante quam integrale per seriem investigemus, nostra aequatio transformari potest in aliam eiusdem formae ponendo

$$z=(a+bx)^m v,$$

unde fit

$$\frac{dz}{z}=\frac{dv}{v}+\frac{mbdx}{a+bx}$$

et

$$\frac{ddz}{z}=\frac{ddv}{v}-\frac{mbbdx^2}{(a+bx)^2}+\frac{2mbdxdv}{v(a+bx)}+\frac{mmbbdx^2}{(a+bx)^2},$$

factaque substitutione

$$\left.\begin{aligned}\frac{(a+bx)ddv}{v}+\frac{2mbdxdv}{v}&+\frac{m(m-1)bbdx^2}{a+bx}\\+\frac{(c+ex)dxdv}{xv}&+\frac{mb(c+ex)dx^2}{x(a+bx)}\\&+\frac{(f+gx)dx^2}{xx}\end{aligned}\right\}=0$$

fiat $(m-1)bx+c+ex$ divisibile per $a+bx$, eritque

$$(m-1)b+e=\frac{bc}{a}\quad\text{et}\quad m=1+\frac{c}{a}-\frac{e}{b},$$

1) In hac formula et plerisque sequentibus partes irrationales tam positive quam negative accipi possunt. Vide finem § 10. H. D.

nostraque aequatio

$$(a+bx)\,ddv+\left(c+\left(\frac{2bc}{a}+2b-e\right)x\right)\frac{dxdv}{x}+\left(f+\left(g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a}\right)x\right)\frac{vdx^2}{xx}=0\,.$$

Ponatur brevitatis gratia

$$\frac{2bc}{a}+2b-e=\varepsilon \quad \text{et} \quad g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a}=\eta\,,$$

ut habeatur forma propositae similis

$$(a+bx)\,ddv+(c+\varepsilon x)\frac{dxdv}{x}+(f+\eta x)\frac{vdx^2}{xx}=0\,,$$

quae ergo est integrabilis, si fuerit

$$\eta=\frac{2abf+c(a\varepsilon-bc)-a(2iiab+(2i+1)(a\varepsilon-bc))\mp(2iab+a\varepsilon-bc)\sqrt{((a-c)^2-4af)}}{2aa}\,;$$

at est

$$a\varepsilon-bc=2ab-ae+bc\,,$$

unde habetur

$$\eta=\frac{2abf+c(2ab-ae+bc)-a(2(i+1)^2ab-(2i+1)\,(ae-bc))\mp(2(i+1)ab-ae+bc)\sqrt{((a-c)^2-4af)}}{2aa}$$

$$=g+\frac{c(ab-ae+bc)}{aa}\,,\ \text{ideoque}$$

$$g=\frac{2abf+c(ae-bc)-a(2(i+1)^2ab-(2i+1)\,(ae-bc))\mp(2(i+1)ab-ae+bc)\sqrt{((a-c)^2-4af)}}{2aa}\,,$$

quae expressio congruit cum praecedente, si ibi loco i ponatur $-i-1$. Quare hic iam pro i omnes numeros integros tam positivos quam negativos sumere licet.

9. Fieri autem potest, ut casus, qui per priorem seriem sunt integrabiles, iidem quoque per posteriorem integrari sicque pro eadem aequatione gemina integralia exhiberi queant. Ponamus enim numerum i pro hac posteriori forma superare numerum integrum i praecedentis formae excessu $\alpha-1$, ita ut hic pro i scribamus $i+\alpha-1$. Quo facto, ut ambo valores ipsius g congruant, fieri necesse est

$$2(i+\alpha)^2ab-(2i+2\alpha-1)\,(ae-bc)=2iiab+(2i+1)\,(ae-bc)$$

et

$$2(i+\alpha)ab-ae+bc=2iab+ae-bc,$$

ex qua sequitur

$$\alpha ab=ae-bc.$$

In priori autem scribendo αab loco $ae-bc$, prodit per ab dividendo

$$2(i+\alpha)^2-2\alpha i-2\alpha\alpha+\alpha=2ii+2\alpha i+\alpha,$$

quae cum sit identica pro omnibus valoribus ipsius i, habebimus

$$\alpha=\frac{ae-bc}{ab},$$

quae expressio debet esse numerus integer.

10. Quoniam igitur infinitos valores pro littera g eruimus, quibus aequatio proposita integrationem admittit, atque adeo formula algebraica pro z satisfaciens assignari potest, operae pretium est, ut hos casus accuratius perpendamus. Denotante ergo i numerum quemcunque integrum sive positivum sive negativum, evolutio prior § 7 facta has duas conditiones postulat:

$$n(n-1)b+ne+g=0$$

et

$$(n-i)(n-i-1)a+(n-i)c+f=0,$$

ex quibus deducitur

$$n=\frac{b-e+\sqrt{((b-e)^2-4bg)}}{2b}$$

et

$$n-i=\frac{a-c+\sqrt{((a-c)^2-4af)}}{2a},$$

unde fit

$$i=\frac{bc-ae+a\sqrt{((b-e)^2-4bg)}-b\sqrt{((a-c)^2-4af)}}{2ab}.$$

Quoties ergo haec formula

$$\frac{bc-ae+a\sqrt{((b-e)^2-4bg)}-b\sqrt{((a-c)^2-4af)}}{2ab},$$

ubi partes irrationales tam positive quam negative accipi possunt, aequatur numero integro sive positivo sive negativo, toties proposita aequatio integrationem admittit.

11. Si coefficientes a, b, c, e, f, g sint rationales, ut hoc fieri possit, vel utrumque signum radicale fieri rationale debet, vel se mutuo destruere. Pro hoc casu fit

$$aa(b-e)^2-4aabg=bb(a-c)^2-4abbf$$

seu

$$4ab(ag-bf)=(ae-bc)(ae+bc-2ab),$$

tum vero fit necesse est

$$i=\frac{bc-ae}{2ab}.$$

Pro illo vero casu si statuamus

$$\sqrt{((a-c)^2-4af)}=h \quad \text{et} \quad \sqrt{((b-e)^2-4bg)}=k,$$

erit

$$f=\frac{(a-c)^2-hh}{4a} \quad \text{et} \quad g=\frac{(b-e)^2-kk}{4b}.$$

Tales igitur valores si habeant litterae f et g, dispiciatur, an haec expressio

$$\frac{bc-ae+ak-bh}{2ab}$$

sit numerus integer. Tum enim si sit numerus integer positivus, valor ipsius z per seriem priorem, sin autem negativus, per posteriorem exhiberi poterit. Ac si insuper

$$\frac{ae-bc}{ab}$$

fuerit numerus integer, utroque modo integratio absolvi poterit, unde integrale completum algebraicum obtinebitur.

12. Casus etiam integrabiles investigari possunt quaerendo factorem[1]), per quem aequatio multiplicata fiat integrabilis. In hunc finem consideremus aequationem huius formae:

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1) Vide notam p. 94 huius voluminis. H. D.

$$ddz + Q\,dx\,dz + R\,z\,dx^2 = 0\,,$$

ita ut sit

$$Q = \frac{c+ex}{x(a+bx)} \quad \text{et} \quad R = \frac{f+gx}{xx(a+bx)}\,.$$

Sitque multiplicator

$$2p\,dz + qz\,dx,$$

ideoque aequatio integrabilis

$$2p\,dz\,ddz + qz\,dx\,ddz + 2pQ\,dx\,dz^2 + Qqz\,dx^2\,dz + 2pRz\,dx^2\,dz + qRzz\,dx^3 = 0.$$

Statuatur aequatio integralis

$$p\,dz^2 + qz\,dx\,dz + zz\,dx^2 \int Rq\,dx = A\,dx^2,$$

cuius differentiali inde ablato fieri debet

$$\left.\begin{aligned} &2Qp\,dx\,dz^2 - dp\,dz^2 - q\,dx\,dz^2 \\ &+ Qqz\,dx^2\,dz + 2Rpz\,dx^2\,dz - z\,dx\,dq\,dz - 2z\,dx^2\,dz\int Rq\,dx \end{aligned}\right\} = 0\,,$$

unde hae duae aequationes existunt:

$$dp + q\,dx = 2Qp\,dx \quad \text{seu} \quad \frac{dp}{p} + \frac{q\,dx}{p} = 2Q\,dx$$

et

$$Qq\,dx + 2Rp\,dx - dq - 2dx\int Rq\,dx = 0.$$

Ponatur

$$\int Rq\,dx = S\,, \quad \text{erit} \quad R\,dx = \frac{dS}{q}$$

et

$$Qq\,dx + \frac{2p\,dS}{q} - dq - 2S\,dx = 0 \quad \text{seu} \quad dS - \frac{Sq\,dx}{p} = \frac{q\,dq}{2p} - \frac{Qqq\,dx}{2p}$$

et pro $Q\,dx$ scripto superiore valore

$$dS - \frac{Sq\,dx}{p} = \frac{q\,dq}{2p} - \frac{qq\,dp}{4pp} - \frac{q^3\,dx}{4pp}\,.$$

Sit $\varkappa$ numerus cuius logarithmus $= 1$, et integrando eruitur

$$\varkappa^{-\int\frac{q\,dx}{p}} S = \int \varkappa^{-\int\frac{q\,dx}{p}} \left(\frac{q\,dq}{2p} - \frac{qq\,dp}{4pp} - \frac{q^3\,dx}{4pp}\right)$$

seu

$$\varkappa^{-\int\frac{q\,dx}{p}} S = \tfrac{1}{4}C + \varkappa^{-\int\frac{q\,dx}{p}} \frac{qq}{4p},$$

unde fit

$$S = \tfrac{1}{4}C\varkappa^{\int\frac{q\,dx}{p}} + \frac{qq}{4p} = \int Rq\,dx\,,$$

hincque porro

$$R = \frac{C\varkappa^{\int\frac{q\,dx}{p}}}{4p} + \frac{2p\,dq - q\,dp}{4pp\,dx} \quad \text{et} \quad Q = \frac{dp}{2p\,dx} + \frac{q}{2p}\,.$$

Ex quibus colligimus, proposita hac aequatione:

$$ddz + \frac{(dp + q\,dx)\,dz}{2p} + (C\varkappa^{\int\frac{q\,dx}{2p}} p\,dx + 2p\,dq - q\,dp)\frac{z\,dx}{4pp} = 0\,,$$

si ea ducatur in $2p\,dz + qz\,dx$, fore integrale

$$p\,dz^2 + qz\,dx\,dz + (C\varkappa^{\int\frac{q\,dx}{p}} p + qq)\frac{zz\,dx^2}{4p} = A\,dx^2\,.$$

13. His ad propositum accommodatis primo obtinemus

$$\frac{(c + ex)\,dx}{x(a + bx)} = \frac{dp}{2p} + \frac{q\,dx}{2p}\,,$$

unde colligimus

$$\int\frac{q\,dx}{p} = -lp + \frac{2c}{a}lx + \frac{2(ae - bc)}{ab}l(a + bx)\,,$$

ideoque

$$\varkappa^{\int\frac{q\,dx}{p}} = \frac{x^{\frac{2c}{a}}(a + bx)^{\frac{2(ae-bc)}{ab}}}{p}$$

hincque

$$\frac{f + gx}{xx(a + bx)} = \frac{Cx^{\frac{2c}{a}}(a + bx)^{\frac{2(ae-bc)}{ab}}dx + 2p\,dq - q\,dp}{4pp\,dx}\,.$$

At est

$$q = \frac{2p(c + ex)}{x(a + bx)} - \frac{dp}{dx}$$

indeque

$$dq = \frac{2(c + ex)\,dp}{x(a + bx)} - \frac{2p\,dx(ac + 2bcx + bexx)}{xx(a + bx)^2} - \frac{ddp}{dx}\,,$$

quibus substitutis aequatio resolvenda erit

$$\frac{4(f+gx)ppdx}{xx(a+bx)}$$
$$=Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}dx+\frac{2(c+ex)pdp}{x(a+bx)}-\frac{4ppdx(ac+2bcx+bexx)}{xx(a+bx)^2}-\frac{2pddp}{dx}+\frac{dp^2}{dx}.$$

14. Verum hoc modo haud minoribus difficultatibus implicamur, quam si ipsam aequationem propositam resolvere vellemus. Aliam ergo viam magis particularem ingrediamur, et quaeramus conditiones coefficientium A, B, C, ut haec aequatio:

$$Ax^\lambda ddz+Bx^{\lambda-1}dxdz+Cx^{\lambda-2}zdx^2=0,$$

si multiplicetur per $2xdz+\alpha zdx$, fiat integrabilis. Cum igitur productum sit

$$\left.\begin{array}{l}2Ax^{\lambda+1}dzddz+2Bx^\lambda dxdz^2+\alpha Bx^{\lambda-1}zdx^2dz+\alpha Cx^{\lambda-2}zzdx^3\\ +\alpha Ax^\lambda zdxddz\qquad\qquad\qquad\qquad +2Cx^{\lambda-1}zdx^2dz\end{array}\right\}=0,$$

integrale sit necesse est[1])

$$Ax^{\lambda+1}dz^2+\alpha Ax^\lambda zdxdz+\frac{\alpha}{\lambda-1}Cx^{\lambda-1}zzdx^2=Edx^2,$$

cuius differentiale si inde auferatur, prodibit haec aequatio:

$$x^\lambda dxdz^2(2B-(\lambda+1)A-\alpha A)+x^{\lambda-1}zdx^2dz\left(\alpha B+2C-\alpha\lambda A-\frac{2\alpha C}{\lambda-1}\right)=0.$$

Unde utroque membro seorsim annihilato fit primo

$$B=\frac{\alpha+\lambda+1}{2}A$$

hincque

$$\frac{\alpha(\alpha-\lambda+1)}{2}A=\frac{2(\alpha-\lambda+1)}{\lambda-1}C,$$

ex qua duplici modo eruitur:

$$\text{vel }\lambda=\alpha+1\quad\text{vel }C=\frac{\alpha(\lambda-1)}{4}A.$$

1) Haec affirmatio quamquam vera est, hic non demonstratur. H. D.

Duae ergo aequationes oriuntur

altera $Ax^{\alpha+1}ddz + (\alpha+1)Ax^{\alpha}dxdz + Cx^{\alpha-1}zdx^2 = 0$,

altera $Ax^{\lambda}ddz + \frac{1}{2}(\alpha+\lambda+1)Ax^{\lambda-1}dxdz + \frac{1}{4}\alpha(\lambda-1)Ax^{\lambda-2}zdx^2 = 0$,

quarum utraque per $2xdz + \alpha zdx$ multiplicata fit integrabilis; illius enim integrale erit

$$Ax^{\alpha+2}dz^2 + \alpha Ax^{\alpha+1}zdxdz + Cx^{\alpha}zzdx^2 = Edx^2,$$

huius vero

$$Ax^{\lambda+1}dz^2 + \alpha Ax^{\lambda}zdxdz + \tfrac{1}{4}\alpha\alpha Ax^{\lambda-1}zzdx^2 = Edx^2.$$

15. Summa ergo harum duarum aequationum eodem factore integrabilis reddetur. Scilicet haec aequatio:

$$\begin{gathered}(Ax^{\alpha+1} + Dx^{\lambda})ddz + ((\alpha+1)Ax^{\alpha} + \tfrac{1}{2}(\alpha+\lambda+1)Dx^{\lambda-1})dxdz \\ + (Cx^{\alpha-1} + \tfrac{1}{4}\alpha(\lambda-1)Dx^{\lambda-2})zdx^2 = 0\end{gathered}$$

multiplicata per $2xdz + \alpha zdx$ integrale praebet

$$(Ax^{\alpha+2}+Dx^{\lambda+1})dz^2+\alpha(Ax^{\alpha+1}+Dx^{\lambda})zdxdz+(Cx^{\alpha}+\tfrac{1}{4}\alpha\alpha Dx^{\lambda-1})zzdx^2=Edx^2,$$

quod isto modo repraesentari potest:

$$(Ax^{\alpha} + Dx^{\lambda-1})\,(xdz + \tfrac{1}{2}\alpha zdx)^2 = dx^2((\tfrac{1}{4}\alpha\alpha A - C)x^{\alpha}zz + E),$$

ita ut sit

$$xdz + \tfrac{1}{2}\alpha zdx = \tfrac{1}{2}dx\sqrt{\frac{4E + (\alpha\alpha A - 4C)x^{\alpha}zz}{Ax^{\alpha} + Dx^{\lambda-1}}}.$$

Ponatur

$$x^{\alpha}zz = vv, \quad \text{erit} \quad x^{\alpha-1}z(2xdz + \alpha zdx) = 2vdv,$$

ideoque

$$2xdz + \alpha zdx = \frac{2vdv}{x^{\alpha-1}z} = \frac{2dv}{x^{\frac{1}{2}\alpha-1}}\ ^{1)};$$

unde fit

$$2x^{1-\frac{1}{2}\alpha}dv = dx\sqrt{\frac{4E + (\alpha\alpha A - 4C)vv}{Ax^{\alpha} + Dx^{\lambda-1}}}$$

seu

1) In editione principe: $x^{\alpha}zz = v$ loco $x^{\alpha}zz = vv$ et $\frac{dv}{x^{\alpha-1}z}$ loco $\frac{2vdv}{x^{\alpha-1}z}$. Correxit H. D.

$$\frac{2dv}{\sqrt{(4E+(\alpha\alpha A-4C)vv)}}=\frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{(Ax^\alpha+Dx^{\lambda-1})}},$$

16. Quo nunc hanc aequationem ad nostram formam perducamus, quod duplici modo fieri potest, ponamus primo $\lambda=\alpha$, ut facta divisione per x^α habeatur haec aequatio:

$$(D+Ax)ddz+((\alpha+\tfrac{1}{2})D+(\alpha+1)Ax)\frac{dxdz}{x}+(\tfrac{1}{4}\alpha(\alpha-1)D+Cx)\frac{zdx^2}{xx}=0,$$

quae multiplicata per $x^\alpha(2xdz+\alpha zdx)$ fit integrabilis, existente integrali posito

$$x^\alpha zz=vv \quad \text{seu} \quad z=x^{-\frac{1}{2}\alpha}v$$

$$\frac{2dv}{\sqrt{(4E+(\alpha\alpha A-4C)vv)}}=\frac{x^{\frac{1}{2}\alpha-1}dx}{\sqrt{x^{\alpha-1}(D+Ax)}}=\frac{dx}{\sqrt{x(D+Ax)}}.$$

Sit iam

$$D=a,\quad A=b,\quad (\alpha+\tfrac{1}{2})a=c,\quad \text{seu}\quad \alpha=\frac{c}{a}-\tfrac{1}{2}\quad \text{et}\quad C=g,$$

et prodibit haec aequatio:

$$(a+bx)ddz+(c+\frac{b(a+2c)}{2a}x)\frac{dxdz}{x}+(\frac{(2c-a)(2c-3a)}{16a}+gx)\frac{zdx^2}{xx}=0,$$

ita ut pro forma proposita sit

$$e=\frac{b(a+2c)}{2a}\quad \text{et}\quad f=\frac{(2c-a)(2c-3a)}{16a},$$

huiusque aequationis posito

$$z=x^{\frac{1}{4}-\frac{c}{2a}}v$$

integrale erit

$$\frac{2dv}{\sqrt{(4E+\left(\frac{b(2c-a)^2}{4aa}-4g\right)vv)}}=\frac{dx}{\sqrt{x(a+bx)}}.$$

17. Statuatur nunc secundo $\lambda=\alpha+2$, ut facta divisione per $x^{\alpha+1}$ oriatur haec aequatio:

$$(A+Dx)ddz+((\alpha+1)A+(\alpha+\tfrac{3}{2})Dx)\frac{dxdz}{x}+(C+\tfrac{1}{4}\alpha(\alpha+1)Dx)\frac{zdx^2}{xx}=0,$$

quae multiplicata per

$$x^{\alpha+1}(2x\,dz + \alpha z\,dx),$$

posito

$$x^{\alpha} zz = vv \quad \text{seu} \quad z = x^{-\frac{1}{2}\alpha} v$$

habebit integrale

$$\frac{2\,dv}{\sqrt{(4E + (\alpha\alpha A - 4C)vv)}} = \frac{x^{\frac{1}{2}\alpha - 1}\,dx}{\sqrt{x^{\alpha}(A + Dx)}} = \frac{dx}{x\sqrt{(A + Dx)}}.$$

Sit iam

$$A = a, \quad D = b, \quad (\alpha + 1)a = c \quad \text{seu} \quad \alpha = \frac{c}{a} - 1 \quad \text{et} \quad C = f,$$

ut obtineatur haec aequatio:

$$(a + bx)\,ddz + \left(c + \frac{b(a + 2c)}{2a}x\right)\frac{dx\,dz}{x} + \left(f + \frac{bc(c - a)}{4aa}x\right)\frac{z\,dx^2}{xx} = 0\,,$$

et pro forma proposita sit

$$e = \frac{b(a + 2c)}{2a} \quad \text{et} \quad g = \frac{bc(c - a)}{4aa},$$

cuius posito

$$z = x^{\frac{1}{2} - \frac{c}{2a}} v$$

integrale est

$$\frac{2\,dv}{\sqrt{\left(4E + \left(\frac{(c - a)^2}{a} - 4f\right)vv\right)}} = \frac{dx}{x\sqrt{(a + bx)}}.$$

18. Non solum autem quoties ipsa aequatio proposita:

$$(a + bx)\,ddz + (c + ex)\frac{dx\,dz}{x} + (f + gx)\frac{z\,dx^2}{xx} = 0$$

in altera harum formarum est contenta, quod evenit, si fuerit

$$\text{vel} \quad e = \frac{b(a + 2c)}{2a} \quad \text{et} \quad f = \frac{(2c - a)(2c - 3a)}{16a}$$

$$\text{vel} \quad e = \frac{b(a + 2c)}{2a} \quad \text{et} \quad g = \frac{bc(c - a)}{4aa},$$

integrationem admittit, sed etiam quoties eadem transformata in alterutra continetur. Transformatio autem ut supra § 8 vidimus fit substitutione

$$z=(a+bx)^{1+\frac{c}{a}-\frac{e}{b}}v,$$

unde oritur

$$(a+bx)ddv+(c+\varepsilon x)\frac{dx\,dv}{x}+(f+\eta x)\frac{v\,dx^2}{xx}=0$$

existente

$$\varepsilon=\frac{2b(a+c)}{a}-e \quad \text{et} \quad \eta=g-\frac{ce}{a}+\frac{bc(a+c)}{aa}.$$

Haec autem ponendo $v=x^n s$, ob

$$\frac{dv}{v}=\frac{n\,dx}{x}+\frac{ds}{s} \quad \text{et} \quad \frac{ddv}{v}=\frac{n(n-1)dx^2}{xx}+\frac{2n\,dx\,ds}{xs}+\frac{dds}{s},$$

transformatur in hanc:

$$\left.\begin{aligned}(a+bx)\frac{dds}{s}+2n(a+bx)\frac{dx\,ds}{xs}&+n(n-1)(a+bx)\frac{dx^2}{xx}\\ +\quad (c+\varepsilon x)\frac{dx\,ds}{xs}&+\qquad n(c+\varepsilon x)\frac{dx^2}{xx}\\ &+\qquad (f+\eta x)\frac{dx^2}{xx}\end{aligned}\right\}=0,$$

unde hi bini casus integrabiles eruuntur.

Primus si

$$D=a,\quad A=b,\quad (\alpha+\tfrac{1}{2})a=2na+c,$$

$$(\alpha+1)b=2nb+\frac{2b(a+c)}{a}-e,$$

$$\tfrac{1}{4}\alpha(\alpha-1)a=n(n-1)a+nc+f,$$

$$C=n(n-1)b+\frac{2nb(a+c)}{a}-ne+g-\frac{ce}{a}+\frac{bc(a+c)}{aa},$$

hincque

$$\alpha=2n-\frac{1}{2}+\frac{c}{a} \quad \text{et} \quad e=\frac{3}{2}b+\frac{bc}{a}=\frac{b(3a+2c)}{2a}$$

atque

$$\left(n-\frac{1}{4}+\frac{c}{2a}\right)\left(n-\frac{3}{4}+\frac{c}{2a}\right)a=n(n-1)a+nc+f,$$

unde

$$f=\frac{cc}{4a}-\frac{c}{2}+\frac{3a}{16}=\frac{(2c-a)(2c-3a)}{16a}.$$

Alter casus his conditionibus continetur:

$$A=a,\quad D=b,\quad (\alpha+1)a=2na-c,\quad (\alpha+\tfrac{3}{2})b=2nb-\frac{2b(a+c)}{a}-e,$$

$$C=n(n-1)a-nc-f,$$

$$\tfrac{1}{4}\alpha(\alpha+1)b=n(n-1)b+\frac{2nb(a-c)}{a}-ne+g-\frac{ce}{a}+\frac{bc(a+c)}{aa},$$

unde fit

$$\alpha=2n-1-\frac{c}{a},\quad e=\frac{b(3a+2c)}{2a}$$

atque

$$g=\frac{bc}{4a}+\frac{bcc}{4aa}=\frac{bc(a+c)}{4aa},$$

ubi constat numerum n nihil conferre.

19. Quatuor ergo hinc nacti sumus casus integrabiles, qui sunt:

$$1^0.\quad e=\frac{b(a+2c)}{2a},\qquad f=\frac{(2c-a)(2c-3a)}{16a}$$

$$2^0.\quad e=\frac{b(a+2c)}{2a},\qquad g=\frac{bc(c-a)}{4aa}$$

$$3^0.\quad e=\frac{b(3a+2c)}{2a},\qquad f=\frac{(2c-a)(2c-3a)}{16a}$$

$$4^0.\quad e=\frac{b(3a+2c)}{2a},\qquad g=\frac{bc(a+c)}{4aa},$$

quibus adeo integrale completum exhibuimus. Videamus ergo quomodo hi casus se habeant ad conditionem, quam supra [§ 10] ex serie deduximus, utrum in ea contineantur nec ne.

Pro primo igitur habemus

$$bc-ae=\frac{-ab}{2},\quad b-e=\frac{b(a-2c)}{2a}$$

et

$$\sqrt{((a-c)^2-4af)}=\pm\frac{a}{2},$$

unde haec formula:

$$-\frac{1}{4}\mp\frac{1}{4}+\frac{\sqrt{(bb(a-2c)^2-16aabg)}}{4ab}$$

numerus integer esse deberet.

Pro secundo est

$$bc-ae=\frac{-ab}{2}, \quad \text{et} \quad \sqrt{((b-e)^2-4bg)}=\pm\frac{b}{2},$$

ergo haec formula:

$$-\frac{1}{4}\pm\frac{1}{4}-\frac{\sqrt{((a-c)^2-4af)}}{2a}$$

numerus integer esse deberet.

Pro tertio est

$$bc-ae=\frac{-3ab}{2}, \quad b-e=\frac{-b(a+2c)}{2a},$$

et

$$\sqrt{((a-c)^2-4af)}=\pm\frac{a}{2},$$

unde haec formula:

$$-\frac{3}{4}\mp\frac{1}{4}+\frac{\sqrt{(bb(a+2c)^2-16aabg)}}{4ab}$$

numerus integer esse deberet.

Pro quarto est

$$bc-ae=\frac{-3ab}{2} \quad \text{et} \quad \sqrt{((b-e)^2-4bg)}=\pm\frac{b}{2},$$

unde haec formula:

$$-\frac{3}{4}\mp\frac{1}{4}-\frac{\sqrt{((a-c)^2-4af)}}{2a}$$

numerus integer esse deberet.

Unde perspicitur hos quatuor casus in superiori conditione non contineri, ideoque hinc omnino novos casus integrabilitatis erui.

20. Cum igitur hi casus, quibus integrale completum eruimus, omnino discrepent ab iis, quibus supra integrale particulare exhibuimus, iuvabit ostendisse, quomodo etiam his casibus integrale completum obtineri possit; quod sequenti modo facillime praestari videtur.

Si aequationi

$$Pddz+Qdxdz+Rzdx^2=0$$

satisfaciat valor $z=V$, ut sit

$$PddV+QdxdV+RVdx^2=0,$$

illa aequatio reddetur integrabilis ducta in

$$\frac{V}{P(Vdz-zdV)}.$$

Posito enim

$$\int\frac{PVddz+QVdxdz+RVzdx^2}{P(Vdz-zdV)}=Sdx,$$

erit

$$Sdx-lP(Vdz-zdV)=\int\frac{QVdxdz+RVzdx^2-VdPdz+PzddV+zdVdP}{P(Vdz-zdV)}$$

$$=\int\frac{Qdx}{P}-lP+\int\frac{z(PddV+QdxdV+RVdx^2)}{P(Vdz-zdV)}.$$

At

$$PddV+QdxdV+RVdx^2=0,$$

ideoque habetur

$$Sdx=l(Vdz-zdV)+\int\frac{Qdx}{P}+\text{Const.}=\text{Const.},$$

unde erit

$$Vdz-zdV=C\varkappa^{-\int\frac{Qdx}{P}}dx$$

et

$$z=CV\int\frac{dx}{VV}\varkappa^{-\int\frac{Qdx}{P}},$$

quod est integrale completum ex particulari $z=V$ erutum.

21. Quoniam utriusque generis casus ex aequatione proposita eliciuntur, si ea per formam $2pdz+qzdx$ multiplicata integrabilis efficiatur, posito $p=uu$, ut sit [§ 12]

$$q=\frac{2uu(c+ex)}{x(a+bx)}-\frac{2udu}{dx},$$

erit aequatio integralis

$$uudz^2+qzdxdz+(C\varkappa^{\int\frac{qdx}{uu}}uu+qq)\frac{zzdx^2}{4uu}=Adx^2,$$

ubi est

$$\varkappa^{\int\frac{qdx}{uu}}=\frac{x^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}}{uu}$$

ideoque

$$\left(udz+\frac{qzdx}{2u}\right)^2=Adx^2-\frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}zzdx^2}{4uu}\text{ [1]},$$

verum quantitatem u ex hac aequatione elici oportet:

$$\frac{ddu}{dx}-\frac{(c+ex)du}{x(a+bx)}+\frac{(f+gx)udx}{xx(a+bx)}+\frac{(ac+2bcx+bexx)udx}{xx(a+bx)^2}=\frac{Cx^{\frac{2c}{a}}(a+bx)^{\frac{2(ae-bc)}{ab}}dx}{4u^2},$$

et prioris quidem generis casus hinc sumta constante $C=0$ sunt deducti. Verum haec aequatio posito

$$u=x^{\frac{c}{a}}(a+bx)^{\frac{ae-bc}{ab}}v$$

abit in hanc:

$$\frac{Cx^{\frac{-2c}{a}}(a+bx)^{\frac{2(bc-ae)}{ab}}dx^2}{4v^2}=ddv+\frac{(c+ex)dxdv}{x(a+bx)}+\frac{(f+gx)vdx^2}{xx(a+bx)},$$

cuius applicatio est facilior, unde si $C=0$, quantitas v satisfacere debet huic aequationi:

$$(a+bx)ddv+\frac{(c+ex)dxdv}{x}+\frac{(f+gx)vdx^2}{xx}=0,$$

ita ut hinc ex valore particulari obtineatur completus. At si ponamus

$$u=x^m(a+bx)^n,$$

erit

$$\tfrac{1}{4}Cx^{\frac{2c}{a}-4m}(a+bx)^{\frac{2(ae-bc)}{ab}-4n}$$

$$=\frac{m(m-1)}{xx}+\frac{(f-mc+(g-me+2mnb)x)}{xx(a+bx)}+\frac{ac+(2-n)bcx+(n-1)(nbb-be)xx}{xx(a+bx)^2},$$

ideoque tam exponentes m et n cum constante C, quam relatio coefficientium a, b, c, e, f, g ex hac aequatione definiri debet:

$$\tfrac{1}{4}Cx^{\frac{2c}{a}-4m+2}(a+bx)^{\frac{2(ae-bc)}{ab}-4n+2}=m(m-1)(a+bx)^2$$

$$+(a+bx)(f-mc+(g-me+2mnb)x)+ac+(2-n)bcx+(n-1)(nb-e)bxx.$$

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1) In editione principe $4u^4$ loco $4u^2$. Correxit H. D.

22. Hinc plures casus resultant, quos evolvamus:

Primus. Si exponens

$$\frac{2(ae-bc)}{ab}-4n+2=2 \quad \text{seu} \quad n=\frac{ae-bc}{2ab},$$

quo esse debet

$$\frac{2c}{a}-4m+2=0 \quad \text{seu} \quad m=\frac{a+c}{2a},$$

ut habeatur

$$\tfrac{1}{4}C(a+bx)^2=\frac{cc-aa}{4aa}(a+bx)^2+(f-\frac{c(a+c)}{2a}+(g-\frac{bc(a+c)}{2aa})x)\,(a+bx)$$
$$+ac+(2-n)bcx+(n-1)\,(nb-e)bxx,$$

ubi postremum membrum per $a+bx$ divisibile esse debet, id quod duplici modo fieri potest.

1°. Vel est $n=1$; ideoque

$$e=\frac{2ab+bc}{a}=\frac{b(2a+c)}{a},$$

sicque erit

$$\tfrac{1}{4}C(a+bx)=\frac{cc-aa}{4aa}(a+bx)+f-\frac{c(a+c)}{2a}+(g-\frac{bc(a+c)}{2aa})x+c,$$

unde fit

$$\tfrac{1}{4}Ca=\frac{cc-aa}{4a}+f+\frac{c(a-c)}{2a}=f-\frac{(a-c)^2}{4a}$$

et

$$\tfrac{1}{4}Cb=\frac{b(cc-aa)}{4aa}+g-\frac{bc(a+c)}{2aa}=g-\frac{b(a+c)^2}{4aa}.$$

Ergo

$$bf-ag-\frac{b(a-c)^2}{4a}+\frac{b(a+c)^2}{4a}=0 \quad \text{seu} \quad g=\frac{bf}{a}+\frac{bc}{a}=\frac{b(c+f)}{a}$$

et

$$\frac{q}{2u}=\frac{u(c+ex)}{x(a+bx)}\frac{du}{dx}=x^{\frac{c-a}{2a}}(c+ex)-\frac{a+c}{2a}x^{\frac{c-a}{2a}}(a+bx)-bx^{\frac{a+c}{2a}}$$

seu

$$\frac{q}{2u}=\frac{a(c-a)+b(a+c)x}{2a}x^{\frac{c-a}{2a}}.$$

Consequenter aequatio integralis

$$(x^{\frac{a+c}{2a}}(a+bx)dz+\frac{a(c-a)+b(a+c)x}{2a}x^{\frac{c-a}{2a}}zdx)^2=A\,dx^2-\left(\frac{f}{a}-\frac{(a-c)^2}{4aa}\right)x^{\frac{c-a}{a}}(a+bx)^2zzdx^2.\,^{1)}$$

2°. Vel est

$$n=\frac{ae-bc}{ab}=\frac{ae-bc}{2ab},$$

ideoque

$$e=\frac{bc}{a}\quad\text{et}\quad n=0;$$

unde fit

$$\tfrac{1}{4}C(a+bx)=\frac{cc-aa}{4aa}(a+bx)+f-\frac{c(a+c)}{2a}+\left(g-\frac{bc(a+c)}{2aa}\right)x+c+\frac{bc}{a}x,$$

ergo

$$\tfrac{1}{4}Ca=\frac{cc-aa}{4a}+f+\frac{c(a-c)}{2a}=f-\frac{(a-c)^2}{4a}$$

et

$$\tfrac{1}{4}Cb=\frac{b(cc-aa)}{4aa}+g+\frac{bc(a-c)}{2aa}=g-\frac{b(a-c)^2}{4aa},$$

unde colligitur

$$bf=ag\quad\text{seu}\quad g=\frac{bf}{a};$$

qui est casus, quo aequatio proposita per $a+bx$ divisibilis existit, sicque nihil habet difficultatis.

23. *Secundus* casus est quo

$$\frac{2(ae-bc)}{ab}-4n+2=1\quad\text{et}\quad\frac{2c}{a}-4m+2=0$$

ideoque

$$m=\frac{a+c}{2a}\quad\text{et}\quad n=\frac{ab+2ae-2bc}{4ab},$$

ita ut habeamus

$$\tfrac{1}{4}C(a+bx)=\frac{cc-aa}{4aa}(a+bx)^2+(f-\frac{c(a+c)}{2a}+(g+\frac{b(a+c)(a-2c)}{4aa})x)(a+bx)$$
$$+ac+(2-n)bcx+(n-1)(nb-e)bxx,$$

qui casus iterum in duos dispertitur:

1) In editione principe $\frac{zzdx^2}{xx}$ loco $x^{\frac{c-a}{a}}(a+bx)^2zzdx^2$. Vide notam p. 163.
Correxit H. D.

1°. Vel est $n = 1$, ideoque

$$2ae - 2bc = 3ab \quad \text{et} \quad e = \frac{3ab + 2bc}{2a} = \frac{b(3a + 2c)}{2a},$$

unde fit

$$\tfrac{1}{4}C = \frac{cc - aa}{4aa}(a + bx) + f + \frac{c(a - c)}{2a} + gx + \frac{b(a + c)(a - 2c)}{4aa}x$$

hincque

$$\tfrac{1}{4}C = \frac{cc - aa}{4a} + f + \frac{c(a - c)}{2a} = f - \frac{(a - c)^2}{4a}$$

et

$$0 = \frac{b(cc - aa)}{4aa} + g + \frac{b(a + c)(a - 2c)}{4aa} \quad \text{seu} \quad g = \frac{bc(a + c)}{4aa},$$

qui est casus quartus in § 19.

2°. Vel est

$$n = \frac{ae - bc}{ab} = \frac{ab + 2ae - 2bc}{4ab},$$

ideoque

$$e = \frac{b(a + 2c)}{2a} \quad \text{et} \quad n = \tfrac{1}{2};$$

unde aequatio per $a + bx$ divisa fit

$$\tfrac{1}{4}C = \frac{cc - aa}{4aa}(a + bx) + f + \frac{c(a - c)}{2a} + \left(g + \frac{b(a + c)(a - 2c)}{4aa}\right)x + \frac{bc}{2a}x,$$

ita ut sit

$$\tfrac{1}{4}C = \frac{cc - aa}{4a} + f + \frac{c(a - c)}{2a} = f - \frac{(a - c)^2}{4a}$$

et

$$\frac{b(cc - aa)}{4aa} + \frac{bc}{2a} + \frac{b(a + c)(a - 2c)}{4aa} + g = 0 \quad \text{seu} \quad g = \frac{bc(c - a)}{4aa},$$

qui erat casus 2° in § 19.

24. *Tertius* casus est, quo

$$\frac{2(ae - bc)}{ab} - 4n + 2 = 1 \quad \text{et} \quad \frac{2c}{a} - 4m + 2 = 1$$

ideoque

$$m = \frac{a + 2c}{4a} \quad \text{et} \quad n = \frac{ab + 2ae - 2bc}{4ab},$$

$$\tfrac{1}{4}Cx(a+bx)=\frac{(a+2c)\,(2c-3a)}{16aa}(a+bx)^2+\Big(f-\frac{c(a+2c)}{4a}+\Big(g+\frac{b(a+2c)\,(a-2c)}{8aa}\Big)x\Big)\,(a+bx)$$
$$+ac+(2-n)bcx+(n-1)\,(nb-e)bxx\,,$$

cuius ultimum membrum duplici modo per $a+bx$ redditur divisibile.

1°. Si $n=1=\frac{ab+2ae-2bc}{4ab}$, ideoque $e=\frac{b(3a+2c)}{2a}$, unde oritur

$$\tfrac{1}{4}Cx=\frac{(a+2c)\,(2c-3a)}{16aa}(a+bx)+f+\frac{c(3a-2c)}{4a}+gx+\frac{b(a+2c)\,(a-2c)}{8aa}x\,,$$

ita ut fieri oporteat

$$\frac{(a+2c)\,(2c-3a)}{16a}+f-\frac{c(2c-3a)}{4a}=0 \quad \text{seu} \quad f=\frac{(2c-a)\,(2c-3a)}{16a}$$

et

$$\tfrac{1}{4}C=\frac{b(a+2c)\,(2c-3a)}{16aa}+g+\frac{b(a+2c)\,(a-2c)}{8aa}=g-\frac{b(a+2c)^2}{16aa}\,,$$

qui erat casus 3° in § 19.

2°. Si $n=\frac{ae-bc}{ab}=\frac{ab+2ae-2bc}{4ab}$ seu $e=\frac{b(a+2c)}{2a}$ et $n=\frac{1}{2}$

$$\tfrac{1}{4}Cx=\frac{(a+2c)\,(2c-3a)}{16aa}(a+bx)+f-\frac{c(a+2c)}{4a}+gx+\frac{b(a+2c)\,(a-2c)}{8aa}x+c+\frac{bc}{2a}x\,,$$

ideoque

$$f+\frac{(a+2c)\,(2c-3a)}{16a}+\frac{c(3a-2c)}{4a}=0 \quad \text{seu} \quad f=\frac{(2c-a)\,(2c-3a)}{16a}$$

et

$$\tfrac{1}{4}C=\frac{b(a+2c)\,(2c-3a)}{16aa}+g+\frac{b(a+2c)\,(a-2c)}{8aa}+\frac{bc}{2a}=g-\frac{b(a-2c)^2}{16aa}\,,$$

qui erat casus 1° in § 19.

25. *Quartus* casus est, quo

$$\frac{2(ae-bc)}{ab}-4n+2=0 \quad \text{et} \quad \frac{2c}{a}-4m+2=0$$

ideoque

$$m = \frac{a+c}{2a} \quad \text{et} \quad n = \frac{ab+ae-bc}{2ab},$$

ut habeamus

$$\tfrac{1}{4}C = \frac{cc-aa}{4aa}(a+bx)^2 + (a+bx)\left(f - \frac{c(a+c)}{2a} + \left(g + \frac{b(aa-cc)}{2aa}\right)x\right)$$
$$+ ac + \frac{3ab-ae+bc}{2a}cx - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx,$$

unde singulas potestates seorsim tollendo colligimus

$$\frac{bb(cc-aa)}{4aa} + bg + \frac{bb(aa-cc)}{2aa} - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa} = 0,$$

$$\frac{b(cc-aa)}{2a} + bf - \frac{bc(a+c)}{2a} + ag + \frac{b(aa-cc)}{2a} + \frac{c(3ab-ae+bc)}{2a} = 0,$$

ex illa fit

$$g = \frac{e(e-2b)}{4b},$$

ex hac vero

$$bf + ag = \frac{c(e-2b)}{2}$$

ideoque

$$f = \frac{(e-2b)(2bc-ae)}{4bb},$$

quae sunt binae conditiones; tum vero capi debebit

$$\tfrac{1}{4}C = \frac{cc-aa}{4} + af - \frac{c(a+c)}{2} + ac = af - \tfrac{1}{4}(a-c)^2.$$

26. *Quintus* casus quo

$$\frac{2(ae-bc)}{ab} - 4n + 2 = 0 \quad \text{et} \quad \frac{2c}{a} - 4m + 2 = 1,$$

ideoque

$$m = \frac{2c+a}{4a} \quad \text{et} \quad n = \frac{ab+ae-bc}{2ab},$$

ut habeamus

$$\tfrac{1}{4}Cx = \frac{(2c+a)(2c-3a)}{16aa}(a+bx)^2 + (a+bx)\left(f - \frac{c(2c+a)}{4a} + \left(g + \frac{b(a-c)(2c+a)}{4aa}\right)x\right)$$
$$+ ac + \frac{3ab-ae+bc}{2a}cx - \frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx$$

hincque

$$\frac{bb(2c+a)(2c-3a)}{16aa}+bg+\frac{bb(a-c)(2c+a)}{4aa}-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}=0$$

seu

$$g=\frac{(b-2e)(3b-2e)}{16b},$$

$$\frac{(2c+a)(2c-3a)}{16}+af-\frac{c(2c+a)}{4}+ac=0 \quad \text{seu} \quad f=\frac{(2c-a)(2c-3a)}{16a}$$

et

$$\tfrac{1}{4}C=\frac{(ab-ae+bc)^2}{4ab}.$$

27. *Sextus* casus quo

$$\frac{2(ae-bc)}{ab}-4n+2=0 \quad \text{et} \quad \frac{2c}{a}-4m+2=2,$$

ideoque

$$m=\frac{c}{2a} \quad \text{et} \quad n=\frac{ab+ae-bc}{2ab},$$

ut habeamus

$$\tfrac{1}{4}Cxx=\frac{c(c-2a)}{4aa}(a+bx)^2+(a+bx)\left(f-\frac{cc}{2a}+\left(g+\frac{bc(a-c)}{2aa}\right)x\right)$$
$$+ac+\frac{3ab-ae+bc}{2a}cx-\frac{(ab-ae+bc)(ab-ae-bc)}{4aa}xx,$$

unde fieri oportet:

$$\frac{c(c-2a)}{4}+af-\frac{cc}{2}+ac=0 \quad \text{seu} \quad f=\frac{c(c-2a)}{4a},$$

$$\frac{bc(c-2a)}{2a}+bf-\frac{bcc}{2a}+ag+\frac{bc(a-c)}{2a}+\frac{3ab-ae+bc}{2a}c=0$$

seu

$$g=\frac{-c(2ab-2ae+bc)}{4aa} \text{ 1)}$$

atque

$$\tfrac{1}{4}C=\frac{-bc(2ab-2ae+bc)}{4aa}-\frac{(b-e)^2}{4}=bg-\tfrac{1}{4}(b-e)^2.$$

1) Editio princeps $4ab$ loco $2ab$. Correxit H. D.

28. Pro his autem casibus omnibus cum sit $u = x^m(a+bx)^n$, erit

$$\frac{q}{2u} = x^{m-1}(a+bx)^{n-1}(c+ex) - mx^{m-1}(a+bx)^n - nbx^m(a+bx)^{n-1}$$

seu

$$\frac{q}{2u} = x^{m-1}(a+bx)^{n-1}(c - ma + (e-(m+n)b)x),$$

unde aequatio integralis colligitur

$$x^{2m}(a+bx)^{2n}\left(dz + \frac{c-ma+(e-(m+n)b)x}{x(a+bx)}zdx\right)^2$$
$$= A\,dx^2 - \tfrac{1}{4}Cx^{\frac{2c}{a}-2m}(a+bx)^{\frac{2(ae-bc)}{ab}-2n}zzdx^2$$

vel erit

$$dz + \frac{c-ma+(e-(m+n)b)x}{x(a+bx)}zdx = \frac{dx\sqrt{\left(A - \tfrac{1}{4}Cx^{\frac{2c}{a}-2m}(a+bx)^{\frac{2(ae-bc)}{ab}-2n}zz\right)}}{x^m(a+bx)^n}.$$

Quare pro casibus inventis integralia aequationis propositae

$$(a+bx)ddz + (c+ex)\frac{dxdz}{x} + \frac{(f+gx)zdx^2}{xx} = 0$$

sequenti modo se habebunt.

CASUS 1

$$m = \frac{a+c}{2a}, \quad n = 1, \quad e = \frac{b(2a+c)}{a}, \quad g = \frac{b(c+f)}{a} \quad \text{et} \quad \tfrac{1}{4}C = \frac{f}{a} - \frac{(a-c)^2}{4aa}.$$

Integrale igitur erit

$$dz + \frac{a(c-a)+b(a+c)x}{2ax(a+bx)}zdx = dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}(a+bx)^2} - \frac{4af-(a-c)^2}{4aaxx}zz\right)}.$$ [1]

1) In editione principe posterius membrum est

$$\frac{dx}{x^{\frac{a+c}{2a}}(a+bx)}\sqrt{\left(A - \frac{(4af-(a-c)^2)}{4aaxx}zz\right)}$$

Vide notam p. 163.

Correxit H. D.

CASUS 2

$$e=\frac{bc}{a},\quad g=\frac{bf}{a},\quad m=\frac{a+c}{2a},\quad n=0 \text{ et } \tfrac{1}{4}C=\frac{4af-(a-c)^2}{4aa}.$$

Integrale ergo erit

$$dz+\frac{a(c-a)+b(c-a)x}{2ax(a+bx)}zdx=dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}}-\frac{4af-(a-c)^2}{4aaxx}zz\right)}\ ^{1)}$$

sive

$$dz+\frac{(c-a)zdx}{2ax}=dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}}+\frac{(a-c)^2-4af}{4aaxx}zz\right)}.$$

CASUS 3

$$e=\frac{b(3a+2c)}{2a},\quad g=\frac{bc(a+c)}{4aa},\quad m=\frac{a+c}{2a},\quad n=1 \text{ et } \tfrac{1}{4}C=\frac{4af-(a-c)^2}{4a},$$

unde integrale est

$$dz+\frac{a(c-a)+bcx}{2ax(a+bx)}zdx=dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}(a+bx)^2}+\frac{((a-c)^2-4af)zz}{4axx(a+bx)}\right)}.$$

CASUS 4

$$e=\frac{b(a+2c)}{2a},\quad g=\frac{bc(c-a)}{4aa},\quad m=\frac{a+c}{2a},\quad n=\tfrac{1}{2} \text{ et } \tfrac{1}{4}C=\frac{4af-(a-c)^2}{4a};$$

unde integrale est

$$dz+\frac{(c-a)zdx}{2ax}=dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}(a+bx)}+\frac{((a-c)^2-4af)zz}{4axx(a+bx)}\right)}.$$

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1) Editio princeps: $\frac{dx}{x^{\frac{a+c}{2a}}}\sqrt{(A\ldots}$ loco $dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}}\ldots\right.}$, casus 2; $\frac{dx}{x^{\frac{a+c}{2a}}(a+bx)}\sqrt{(A\ldots}$ loco $dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}(a+bx)^2}\ldots\right.}$, casus 3; $\frac{dz}{x^{\frac{a+c}{2a}}(a+bx)^{\frac{1}{2}}}\sqrt{(A\ldots}$ loco $dx\sqrt{\left(\frac{A}{x^{\frac{a+c}{a}}(a+bx)}\ldots\right.}$, casus 4; $\frac{dx}{x^{\frac{a+2c}{4a}}(a+bx)}\sqrt{(A\ldots}$ loco $dx\sqrt{\left(\frac{A}{x^{\frac{a+2c}{2a}}(a+bx)^2}\ldots\right.}$, casus 5; $\frac{dx}{x^{\frac{a+2c}{4a}}(a+bx)^{\frac{1}{2}}}\sqrt{(A\ldots}$ loco $dx\sqrt{\left(\frac{A}{x^{\frac{a+2c}{2a}}(a+bx)}\ldots\right.}$, casus 6; $\frac{dx}{x^m(a+bx)^n}\sqrt{(A\ldots}$ loco $dx\sqrt{\left(\frac{A}{[illegible]}\ldots\right.}$, casus 7, 8 et 9. Vide notam 1) p. 170.

CASUS 5.

$$e=\frac{b(3a+2c)}{2a},\quad f=\frac{(2c-a)(2c-3a)}{16a},\quad m=\frac{a+2c}{4a},\quad n=1 \text{ et } \tfrac{1}{4}C=g-\frac{b(a+2c)^2}{16aa},$$

unde integrale erit

$$dz+\frac{a(2c-a)+b(2c+a)x}{4ax(a+bx)}zdx=dx\sqrt{\left(\frac{A}{x^{\frac{a+2c}{a}}(a+bx)^2}+\frac{(b(a+2c)^2-16aag)zz}{16aax(a+bx)}\right)}.$$

CASUS 6

$$e=\frac{b(a+2c)}{2a},\quad f=\frac{(2c-a)(2c-3a)}{16a},\quad m=\frac{a+2c}{4a},\quad n=\tfrac{1}{2} \text{ et } \tfrac{1}{4}C=g-\frac{b(a-2c)^2}{16aa},$$

unde integrale erit

$$dz+\frac{(2c-a)zdx}{4ax}=dx\sqrt{\left(\frac{A}{x^{\frac{a+2c}{a}}(a+bx)}+\frac{(b(a+2c)^2-16aag)zz}{16aax(a+bx)}\right)}.$$

CASUS 7

$$f=\frac{(e-2b)(2bc-ae)}{4bb},\quad g=\frac{e(e-2b)}{4b},\quad m=\frac{a+c}{2a},\quad n=\frac{ab+ae-bc}{2ab} \text{ et } \tfrac{1}{4}C=af-\tfrac{1}{4}(a-c)^2,$$

unde integrale erit

$$dz+\frac{c-a+(e-2b)x}{2x(a+bx)}zdx=dx\sqrt{\left(\frac{A}{x^{2m}(a+bx)^{2n}}+\frac{((a-c)^2-4af)zz}{4xx(a+bx)^2}\right)}.$$

CASUS 8

$$f=\frac{(2c-a)(2c-3a)}{16a},\quad g=\frac{(b-2e)(3b-2e)}{16b},\quad m=\frac{2c+a}{4a},\quad n=\frac{ab+ae-bc}{2ab} \text{ et } \tfrac{1}{4}C=\frac{(ab-ae+bc)^2}{4ab};$$

unde integrale erit

$$dz+\frac{2c-a+(2e-3b)x}{4x(a+bx)}zdx=dx\sqrt{\left(\frac{A}{x^{2m}(a+bx)^{2n}}-\frac{(ab-ae+bc)^2zz}{4abx(a+bx)^2}\right)}.$$

CASUS 9

$$f=\frac{c(c-2a)}{4a},\quad g=\frac{-c(2ab-2ae+bc)}{4aa}\text{[1]},\quad m=\frac{c}{2a},\quad n=\frac{ab+ae-bc}{2ab}\quad\text{et}\quad \tfrac{1}{4}C=bg-\tfrac{1}{4}(b-e)^2,$$

unde integrale

$$dz+\frac{c+(e-b)x}{2x(a+bx)}zdx=dx\sqrt{\left(\frac{A}{x^{2m}(a+bx)^{2n}}+\frac{((b-e)^2-4bg)zz}{4(a+bx)^2}\right)}.$$

29. Praeter hos vero novem casus, quibus binae relationes inter coefficientes praescribuntur, initio innumerabiles casus integrabiles duplici modo eruimus. Altero priori § 6 integrale algebraicum huius formae:

$$z=Ax^n+Bx^{n+1}+Cx^{n+2}+Dx^{n+3}+\text{etc.}$$

assignari potest, si denotante i numerum integrum positivum quemcunque fuerit

$$n(n-1)a+nc+f=0$$

et

$$(n+i)(n+i-1)b+(n+i)e+g=0.$$

Altero vero posteriori § 8 integrale huius est formae:

$$z=(a+bx)^{\frac{ab-ae+bc}{ab}}(Ax^n+Bx^{n+1}+Cx^{n+2}+\text{etc.}),$$

si fuerit

$$n(n-1)a+nc+f=0$$

et

$$(n+i)(n+i-1)b+(n+i)\left(\frac{2bc}{a}+2b-e\right)+g+\frac{bc}{a}+\frac{bcc}{aa}-\frac{ce}{a}=0.$$

Quae integralia etsi sunt particularia, tamen ex iis completa facile determinantur.

1) Editio princeps $4ab$ loco $2ab$. Vide notam p. 169. Correxit H. D.

SUMMATIO FRACTIONIS CONTINUAE CUIUS INDICES PROGRESSIONEM ARITHMETICAM CONSTITUUNT DUM NUMERATORES OMNES SUNT UNITATES UBI SIMUL RESOLUTIO AEQUATIONIS RICCATIANAE PER HUIUSMODI FRACTIONES DOCETUR

Conventui exhibita die 18. septembris 1775
Commentatio 595 indicis ENESTROEMIANI
Opuscula analytica 2, 1785, p. 217—239

1. Cum in praecedente dissertatione[1]) methodum exposuissem, fractiones continuas ad duas formulas integrales reducendi, ea quidem infinitis casibus feliciter successit: at vero casus, qui simplicissimus videtur, ubi omnes numeratores inter se ponuntur aequales, ad eiusmodi formulas integrales perduxit, quas nullo adhuc modo evolvere et inter se comparare licuit, cum tamen ex hoc genere binae fractiones continuae habeantur, quarum valores satis commode exhiberi possunt:

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\text{etc.}}}}=\frac{e^{\frac{2}{n}}+1}{e^{\frac{2}{n}}-1}$$

et

1) L. EULERI Commentatio 594 (indicis *Enestroemiani*): *Methodus inveniendi formulas integrales, quae certis casibus inter se teneant rationem, ubi simul methodus traditur fractiones continuas summandi.* Opuscula analytica 2, 1785, p. 178. *LEONHARDI EULERI Opera omnia*, series I, vol. 18, p. 209. H. D.

$$n - \cfrac{1}{3n - \cfrac{1}{5n - \cfrac{1}{7n - \text{etc.}}}} = \cot.\frac{1}{n},$$

quarum quidem altera ex altera facile deducitur, si loco n scribatur $n\sqrt{-1}$.

2. Quando autem indices aliam quamcunque progressionem arithmeticam sequuntur, summationem talium fractionum continuarum iam olim in Tomo XI veterum Commentariorum nostrae Academiae[1]) singulari prorsus modo ad aequationem RICCATIANAM reduxi. Methodus autem, qua hoc praestiti, ibi nimis succincte est exposita; quare, cum ea plurimum in recessu habere videatur, eam hic operae pretium erit uberius explicare; praecipue cum non solum nemo vim illius methodi animadvertisse videatur, sed etiam ipse eius penitus essem oblitus.

3. Quo isthanc investigationem clarius ob oculos ponam, exordiar a fractione generali, cuius quidem omnes numeratores sint unitates, indices autem in genere litteris a, b, c, d, e, f etc. designentur, ita ut ipsa fractio continua hanc habeat formam:

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

cuius valor littera S indicetur, ad quem proxime saltem cognoscendum, ex indicibus a, b, c, d, e etc. formetur more solito series sequentium fractionum:

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{A}{\mathfrak{A}} & \frac{B}{\mathfrak{B}} & \frac{C}{\mathfrak{C}} & \frac{D}{\mathfrak{D}} & \frac{E}{\mathfrak{E}} & \text{etc.,} \end{array}$$

quarum tam numeratores quam denominatores sequenti modo ex binis praecedentibus determinantur:

1) L. EULERI Commentatio 123 (indicis Enestroemiani): *De fractionibus continuis observationes.* Comment. acad. sc. Petrop. 11 (1750), p. 32/81. *LEONHARDI EULERI Opera omnia*, series I, vol. 14, p. 291. Cf. quoque Commentationem 751 huius voluminis. H. D.

$$A = a,\ B = Ab + 1,\ C = Bc + A,\ D = Cd + B \text{ etc.}$$
$$\mathfrak{A} = 1,\ \mathfrak{B} = \mathfrak{A}b,\quad \mathfrak{C} = \mathfrak{B}c + \mathfrak{A},\ \mathfrak{D} = \mathfrak{C}d + \mathfrak{B} \text{ etc.}$$

4. Nunc autem loco litterarum $A, B, C, D \ldots$ et $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D} \ldots$ alias in calculum introducamus, quae sint

$$A' = \frac{A}{a},\quad B' = \frac{B}{ab},\quad C' = \frac{C}{abc},\quad D' = \frac{D}{abcd} \text{ etc.}$$
$$\mathfrak{A}' = \frac{\mathfrak{A}}{a},\quad \mathfrak{B}' = \frac{\mathfrak{B}}{ab},\quad \mathfrak{C}' = \frac{\mathfrak{C}}{abc},\quad \mathfrak{D}' = \frac{\mathfrak{D}}{abcd} \text{ etc.,}$$

atque hae novae litterae sequenti modo per indices et binos antecedentes terminos determinabuntur:

$$A' = 1,\quad B' = A' + \frac{1}{ab},\quad C' = B' + \frac{A'}{bc},\quad D' = C' + \frac{B'}{cd} \text{ etc.}$$
$$\mathfrak{A}' = \frac{1}{a},\quad \mathfrak{B}' = \mathfrak{A}',\qquad \mathfrak{C}' = \mathfrak{B}' + \frac{\mathfrak{A}'}{bc},\quad \mathfrak{D}' = \mathfrak{C}' + \frac{\mathfrak{B}'}{cd} \text{ etc.}$$

His igitur valoribus pro quovis casu evolutis, istae fractiones:

$$\frac{A'}{\mathfrak{A}'},\quad \frac{B'}{\mathfrak{B}'},\quad \frac{C'}{\mathfrak{C}'},\quad \frac{D'}{\mathfrak{D}'},\quad \frac{E'}{\mathfrak{E}'} \text{ etc.}$$

continuo propius ad valorem S fractionis continuae propositae accedent, et in infinitum continuatae ei prorsus aequabuntur.

5. Quo formae harum litterarum melius perspiciantur, eas simpliciter per indices evolvamus, ac primo quidem pro numeratoribus reperiemus sequentes formulas:

$$A' = 1$$
$$B' = 1 + \frac{1}{ab}$$
$$C' = 1 + \frac{1}{ab} + \frac{1}{bc}$$
$$D' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{abcd}$$
$$E' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde}$$
$$F' = 1 + \frac{1}{ab} + \frac{1}{bc} + \frac{1}{cd} + \frac{1}{de} + \frac{1}{ef} + \frac{1}{abcd} + \frac{1}{abde} + \frac{1}{bcde} + \frac{1}{abef} + \frac{1}{bcef} + \frac{1}{cdef} + \frac{1}{abcdef}$$
etc.

Pro denominatoribus vero prodibunt sequentes formulae:

$$\mathfrak{A}' = \frac{1}{a}$$

$$\mathfrak{B}' = \frac{1}{a}$$

$$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$$

$$\mathfrak{D}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd}$$

$$\mathfrak{E}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{abcde}$$

$$\mathfrak{F}' = \frac{1}{a} + \frac{1}{abc} + \frac{1}{acd} + \frac{1}{ade} + \frac{1}{aef} + \frac{1}{abcde} + \frac{1}{abcef} + \frac{1}{acdef}$$

etc.,

in quibus posterioribus formulis singuli termini factorem habent $\frac{1}{a}$, quo omisso reliqui factores eodem modo per indices b, c, d, e, f etc. definientur, quo litterae latinae antecedentes per omnes indices a, b, c, d, e etc. sunt determinatae.

6. Accommodemus nunc istas evolutiones ad casum, qui hic nobis est propositus, ubi indices a, b, c, d, e secundum progressionem arithmeticam procedere assumimus. Statuamus igitur differentiam, qua hi indices continuo crescunt $= \Delta$, eruntque indices post primum sequentes

$$b = a + \Delta, \quad c = a + 2\Delta, \quad d = a + 3\Delta, \quad e = a + 4\Delta \quad \text{etc.}$$

Hos quidem valores in denominatoribus nostrarum formularum non substituemus, sed iis praecipue utemur ad formulas contrahendas.

7. Hac igitur progressione stabilita evolvamus primo numeratores nostrarum fractionum sequenti modo:

$$A' = 1$$

$$B' = 1 + \frac{1}{ab}$$

$$C' = 1 + \frac{1}{ab} + \frac{1}{bc} = 1 + \frac{2}{ac}$$

$$D' = 1 + \frac{2}{ac} + \frac{1}{cd} + \frac{1}{abcd} = 1 + \frac{3}{ad} + \frac{1}{abcd}$$

$$E' = 1 + \frac{3}{ad} + \frac{1}{de} + \frac{1}{abcd} + \frac{2}{acde} = 1 + \frac{4}{ae} + \frac{3}{abde}$$

$$F' = 1 + \frac{5}{af} + \frac{6}{abdf} + \frac{1}{abcdef}$$

$$G' = 1 + \frac{6}{ag} + \frac{10}{abfg} + \frac{4}{abcefg}$$

$$H' = 1 + \frac{7}{ah} + \frac{15}{abgh} + \frac{10}{abcfgh} + \frac{1}{abcdefgh}$$

$$I' = 1 + \frac{8}{ai} + \frac{21}{abhi} + \frac{20}{abcghi} + \frac{5}{abcdfghi}$$

$$K' = 1 + \frac{9}{ak} + \frac{28}{abik} + \frac{35}{abchik} + \frac{15}{abcdghik} + \frac{1}{abcdefghik}$$

etc.

8. Quemadmodum in his formis omnes termini primi sunt unitates, ita numeratores secundorum secundum numeros naturales, tertiorum secundum trigonales, quartorum secundum pyramidales primos, tum secundos, tertios etc. progrediuntur. In denominatoribus ordo pariter satis est manifestus. Hinc igitur in genere eam formulam exhibere poterimus, quae indefinite respondeat indici i. Ita si ista expressio designetur littera Z', tum vero in ordine litterarum a, b, c, d, e etc. fuerit $z = a + i\Delta$, antecedentes vero

$$y = a + (i-1)\Delta, \quad x = a + (i-2)\Delta, \quad v = a + (i-3)\Delta \quad \text{etc.},$$

habebimus

$$Z' = 1 + \frac{i}{az} + \frac{(i-1)(i-2)}{1\cdot 2\cdot abyz} + \frac{(i-2)(i-3)(i-4)}{1\cdot 2\cdot 3\cdot abcxyz} + \frac{(i-3)(i-4)(i-5)(i-6)}{1\cdot 2\cdot 3\cdot 4\cdot abcdvxyz} + \text{ etc.}$$

9. Quod iam ad litteras germanicas $\mathfrak{A}'$, $\mathfrak{B}'$, $\mathfrak{C}'$, $\mathfrak{D}'$ etc. attinet, earum quaelibet ex antecedente latina formatur, dum litterae a, b, c, d, e uno gradu promoventur, tum vero singuli termini per $\frac{1}{a}$ multiplicentur, ita erit ut sequitur:

$$\mathfrak{A}' = 1$$

$$\mathfrak{B}' = \frac{1}{a}$$

$$\mathfrak{C}' = \frac{1}{a} + \frac{1}{abc}$$

$$\mathfrak{D}' = \frac{1}{a} + \frac{2}{abd}$$

$$\mathfrak{E}' = \frac{1}{a} + \frac{3}{abe} + \frac{1}{abcde}$$

$$\mathfrak{F}' = \frac{1}{a} + \frac{4}{abf} + \frac{3}{abcef}$$

$$\mathfrak{G}' = \frac{1}{a} + \frac{5}{abg} + \frac{6}{abcfg} + \frac{1}{abcdefg}$$

$$\mathfrak{H}' = \frac{1}{a} + \frac{6}{abh} + \frac{10}{abcgh} + \frac{4}{abcdfgh}$$

$$\mathfrak{I}' = \frac{1}{a} + \frac{7}{abi} + \frac{15}{abchi} + \frac{10}{abcdghi} + \frac{1}{abcdefghi}$$

etc.,

unde in genere, si indici i respondeat $\mathfrak{Z}'$, erit

$$\mathfrak{Z}' = \frac{1}{a} + \frac{(i-1)}{abz} + \frac{(i-2)(i-3)}{1\cdot 2\cdot abcyz} + \frac{(i-3)(i-4)(i-5)}{1\cdot 2\cdot 3\cdot abcdxyz} + \frac{(i-4)(i-5)(i-6)(i-7)}{1\cdot 2\cdot 3\cdot 4\cdot abcdevxyz} + \text{etc.}$$

10. Augeamus nunc indicem z in infinitum, ut fractio $\frac{Z'}{\mathfrak{Z}'}$ exprimat ipsum valorem fractionis continuae, quem vocavimus S, ita ut futurum sit $S = \frac{Z'}{\mathfrak{Z}'}$, atque reductio formularum inventarum sequenti modo peragetur. Primo scilicet pro Z' erit

$$\frac{i}{z} = \frac{i}{a+i\Delta} = \frac{1}{\Delta}, \quad \text{ob } i = \infty;$$

pro termino tertio erit

$$\frac{(i-1)(i-2)}{yz} = \frac{(i-1)(i-2)}{(a+(i-1)\Delta)(a+i\Delta)} = \frac{1}{\Delta^2};$$

porro autem simili modo

$$\frac{(i-2)(i-3)(i-4)}{xyz} = \frac{(i-2)(i-3)(i-4)}{(a+(i-2)\Delta)(a+(i-1)\Delta)(a+i\Delta)} = \frac{1}{\Delta^3}$$

et

$$\frac{(i-3)(i-4)(i-5)(i-6)}{vxyz}=\frac{1}{\varDelta^4} \quad \text{etc.}$$

Pari modo pro formula $\mathfrak{Z}'$ erit etiam

$$\frac{i-1}{z}=\frac{1}{\varDelta}, \quad \frac{(i-2)(i-3)}{yz}=\frac{1}{\varDelta^2}$$

et ita porro; quamobrem pro casu $i=\infty$ ambae nostrae formulae ita commode contrahuntur, ut fiat

$$Z'=1+\frac{1}{a\varDelta}+\frac{1}{1\cdot 2ab\varDelta^2}+\frac{1}{1\cdot 2\cdot 3abc\varDelta^3}+\frac{1}{1\cdot 2\cdot 3\cdot 4abcd\varDelta^4}+\text{ etc.}$$

similique modo

$$\mathfrak{Z}'=\frac{1}{a}+\frac{1}{ab\varDelta}+\frac{1}{1\cdot 2abc\varDelta^2}+\frac{1}{1\cdot 2\cdot 3abcd\varDelta^3}+\frac{1}{1\cdot 2\cdot 3\cdot 4abcde\varDelta^4}+\text{ etc.}$$

11. Cum igitur sit valor quaesitus $S=\frac{Z'}{\mathfrak{Z}'}$, videamus, quomodo ambas series infinitas inventas ad expressiones finitas revocare queamus. Hunc in finem ambas series generaliores reddamus, dum loco numeratorum, qui omnes sunt 1, progressionem quandam geometricam substituimus. Statuamus igitur

$$p=1+\frac{x^{\varDelta}}{a\varDelta}+\frac{x^{2\varDelta}}{1\cdot 2ab\varDelta^2}+\frac{x^{3\varDelta}}{1\cdot 2\cdot 3abc\varDelta^3}+\text{ etc.}$$

et

$$q=\frac{1}{a}+\frac{x^{\varDelta}}{ab\varDelta}+\frac{x^{2\varDelta}}{1\cdot 2abc\varDelta^2}+\frac{x^{3\varDelta}}{1\cdot 2\cdot 3abcd\varDelta^3}+\text{ etc.}$$

Quod si autem hinc valores p et q eruerimus in genere, tum vero ponamus $x=1$, utique proveniet $S=\frac{p}{q}$. Hic autem manifestum est, ambas has series egregiam inter se tenere affinitatem, ac per differentiationem unam in alteram converti posse, quam investigationem sequenti modo instituamus.

12. Primo igitur series prior simpliciter differentiata dat

$$\frac{x\,dp}{dx}=\frac{x^{\varDelta}}{a}+\frac{x^{2\varDelta}}{ab\varDelta}+\frac{x^{3\varDelta}}{1\cdot 2abc\varDelta^2}+\frac{x^{4\varDelta}}{1\cdot 2\cdot 3abcd\varDelta^3}+\text{ etc.,}$$

es cum altera q comparata manifesto praebet

$$\frac{xdp}{dx} = x^{\Delta} q\,,$$

unde patet, si modo summa alterius harum duarum serierum esset cognita, alteram quoque assignari posse, quandoquidem ex cognito valore p prodit

$$q = \frac{dp}{x^{\Delta-1}\,dx}\,;$$

contra vero ex cognito valore q fit

$$dp = x^{\Delta-1} q\,dx,$$

ideoque

$$p = \int x^{\Delta-1} q\,dx,$$

quod integrale ita sumi debet, ut posito $x = 0$ fiat $p = 1$.

13. Ante autem quam alteram seriem differentiemus, eam multiplicemus per x^a, atque ob

$$a + \Delta = b,\ a + 2\Delta = c,\ a + 3\Delta = d \text{ etc.}$$

erit

$$x^a q = \frac{x^a}{a} + \frac{x^b}{ab\Delta} + \frac{x^c}{1\cdot 2abc\Delta^2} + \frac{x^d}{1\cdot 2\cdot 3abcd\Delta^3} + \text{ etc.}$$

Nunc ista aequatio differentiata iterumque per x multiplicata dabit

$$x d\cdot\frac{x^a q}{dx} = x^a + \frac{x^b}{a\Delta} + \frac{x^c}{1\cdot 2ab\Delta^2} + \frac{x^d}{1\cdot 2\cdot 3abc\Delta^3} + \text{ etc.},$$

at vero prior series p itidem per x^a multiplicata praebet

$$x^a p = x^a + \frac{x^b}{a\Delta} + \frac{x^c}{1\cdot 2ab\Delta^2} + \frac{x^d}{1\cdot 2\cdot 3abc\Delta^3} + \text{ etc.}\,,$$

quae series cum sint perfecte aequales, erit

$$\frac{x}{dx} d\cdot x^a q = x^a p\,, \quad \text{ideoque} \quad d\cdot x^a q = p x^{a-1} dx\,,$$

sicque duas nacti sumus aequationes differentiales inter p et q, ex quibus valorem utriusque eruere licebit.

14. Cum ex priore aequatione sit

$$q = \frac{dp}{x^{\Delta-1}dx}, \quad \text{erit} \quad x^a q = \frac{x^{a-\Delta+1}dp}{dx},$$

unde sumto elemento dx constante fiet

$$d\cdot x^a q = \frac{x^{a-\Delta+1}ddp + (a-\Delta+1)x^{a-\Delta}dp\,dx}{dx},$$

sicque elisa quantitate q pro altera p nanciscimur hanc aequationem differentialem secundi gradus:

$$px^{a-1}dx = \frac{x^{a-\Delta+1}ddp + (a-\Delta+1)x^{a-\Delta}dp\,dx}{dx},$$

quam si resolvere licuerit, totum negotium erit confectum. Ubi probe notandum, cum sit

$$p = 1 + \frac{x^\Delta}{a\Delta} + \frac{x^{2\Delta}}{1\cdot 2ab\Delta^2} + \text{ etc.},$$

integrationem ita institui debere, ut posito $x=0$ fiat $p=1$; tum vero, quia est

$$\frac{dp}{dx} = \frac{x^{\Delta-1}}{a} + \frac{x^{2\Delta-1}}{ab\Delta} + \text{ etc.},$$

altera conditio integrationis postulat, ut posito $x=0$ etiam fiat $\frac{dp}{dx}=0$, siquidem fuerit $\Delta>1$; si enim esset $=1$, casu $x=0$ fieri debet $\frac{dp}{dx}=\frac{1}{a}$. At si $\Delta<1$, fieri debebit $\frac{dp}{dx}=\infty$.

15. Cum fractio $\frac{p}{q}$ posito $x=1$ praebeat valorem nostrae fractionis continuae S, ponamus in genere esse $\frac{p}{q}=z$, ita ut posito $x=1$ fiat $z=S$, unde patet, casu $x=0$ fieri debere $z=a$. Cum igitur sit $p=qz$, erit $dp=q\,dz+z\,dq$; erat autem ex prima aequatione

$$q = \frac{dp}{x^{\Delta-1}dx},$$

quamobrem habebimus

$$q = \frac{q\,dz + z\,dq}{x^{\Delta-1}dx},$$

sive $x^{\Delta-1}q\,dx = q\,dz + z\,dq$, unde fit

$$dq = \frac{x^{\Delta-1}q\,dx - q\,dz}{z};$$

erat autem $dp = x^{\Delta-1}q\,dx$.

16. Haec igitur sequuntur ex priori aequatione differentiali inventa $\frac{dp}{x^{\Delta-1}dx} = q$. Altera vero, quae est

$$d\cdot x^a q = p x^{a-1}dx,$$

quia

$$d\cdot x^a q = a x^{a-1} q\,dx + x^a\,dq$$

loco dq posito valore modo invento prodit

$$d\cdot x^a q = a x^{a-1} q\,dx + \frac{x^{a+\Delta-1} q\,dx - x^a q\,dz}{z},$$

habebimus

$$p x^{a-1}dx = a x^{a-1} q\,dx + \frac{x^{a+\Delta-1} q\,dx - x^a q\,dz}{z},$$

quae aequatio, ob $p = qz$, per q divisa suppeditat istam aequationem differentialem primi gradus:

$$x^{a-1} z\,dx = a x^{a-1}dx + \frac{x^{a+\Delta-1}dx - x^a\,dz}{z},$$

quae per z multiplicata et per x^{a-1} divisa praebet

$$zz\,dx = az\,dx + x^{\Delta}dx - x\,dz,$$

cuius ergo resolutio si ita instituatur, ut posito $x = 0$ fiat $z = a$, tum vero fiat $x = 1$, valor ipsius z dabit ipsum valorem fractionis continuae quem quaerimus.

17. Totum igitur negotium perductum est ad resolutionem aequationis differentialis primi gradus

$$zz\,dx = az\,dx + x^{\Delta}dx - x\,dz,$$

quae manifesto in celeberrima illa aequatione RICCATIANA continetur. Ut enim ad tres tantum terminos reducatur, ponatur $z = x^a y$, ita ut sit $y = \frac{z}{x^a}$, unde fit

$$dz = ax^{a-1}y\,dx + x^a\,dy,$$

quibus valoribus substitutis nostra aequatio hanc induet formam:

$$x^{a+1}dy + x^{2a}yy\,dx = x^{\varDelta}dx,$$

quam ergo ita integrari oportet, ut posito $x = 0$, sive infinite parvo, fiat $y = \frac{a}{x^a}$, hoc est $y = \infty$, quo facto, si post integrationem statuatur $x = 1$, valor ipsius y dabit summam fractionis continuae propositae.

18. Quo hanc expressionem simpliciorem reddamus, dividamus per x^{a+1}, ut habeamus

$$dy + x^{a-1}yy\,dx = x^{\varDelta-a-1}dx;$$

nunc vero statuamus $x^a = t$, ita ut casu $x = 0$ fiat quoque $t = 0$, et casu $x = 1$ etiam $t = 1$, unde si t evanescat, fieri debet $y = \frac{a}{t}$, sive $y = \infty$. Hoc autem valore introducto, ob $x = t^{\frac{1}{a}}$ et

$$dx = \frac{t^{\frac{1-a}{a}}dt}{a},$$

aequatio nostra fiet

$$a\,dy + yy\,dt = t^{\frac{\varDelta-2a}{a}}dt,$$

quae est forma maxime usitata aequationis RICCATIANAE.

19. Quando ergo proposita fuerit talis fractio continua[1]):

$$a + \cfrac{1}{a + \varDelta + \cfrac{1}{a + 2\varDelta + \cfrac{1}{a + 3\varDelta + \cfrac{1}{a + 4\varDelta + \text{etc.}}}}}$$

1) Cf. Commentationem 751 huius voluminis § 20, p. 425. H. D.

pro eius valore investigando resolvi debet ista aequatio RICCATIANA:

$$ady + yydt = t^{\frac{\Delta - 2a}{a}} dt\,;$$

ubi integrationem ita institui oportet, ut sumta t infinite parva fiat $y = \frac{a}{t}$, quo facto statuatur $t = 1$, et valor pro y resultans erit valor huius fractionis continuae.

20. Evolvamus casum simplicissimum, quo ad dextram partem exponens ipsius t fit nihilo aequalis, quod ergo evenit, si $\Delta = 2a$, ideoque ipsa fractio continua

$$a + \cfrac{1}{3a + \cfrac{1}{5a + \cfrac{1}{7a + \text{etc.}}}}$$

pro cuius summa habebimus hanc aequationem differentialem:

$$ady + yydt = dt\,, \quad \text{unde} \quad dt = \frac{ady}{1 - yy}\,,$$

et integrando

$$t = \frac{a}{2} l \frac{1+y}{1-y} + C\,,$$

quae constans C ita est capienda, ut posito $t = 0$ fiat $y = \frac{a}{t}$, ideoque $t = \frac{a}{y}$; unde patet hoc casu fieri $y = \infty$, ex quo statim intelligitur, aequationem integralem ita instrui debere:

$$t = \frac{a}{2} l \frac{y+1}{y-1} + C\,,$$

sive integrationem ita institui, ut facto $t = 0$ fiat y infinitum. Nunc vero si y infinitum, erit $l\frac{y+1}{y-1} = \frac{2}{y}$, quare, cum fieri debeat $t = \frac{a}{y}$, fit $C = 0$, ita ut iusta aequatio integralis sit $t = \frac{a}{2} l \frac{y+1}{y-1}$, unde denotante e numerum, cuius logarithmus hyperbolicus $= 1$, fiet $e^{\frac{2t}{a}} = \frac{y+1}{y-1}$, hincque porro $y = \frac{e^{\frac{2t}{a}}+1}{e^{\frac{2t}{a}}-1}$,

unde posito $t=1$ summa nostrae fractionis continuae erit $\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1}$, qui est idem valor, quem iam dudum inveneram.[1])

21. Contemplemur nunc etiam reliquos casus integrabilitatis aequationis RICCATIANAE, quibus exponens ipsius t ad partem dextram est vel -4, vel $-\frac{4}{3}$, vel $-\frac{8}{3}$, vel $-\frac{8}{5}$, vel $-\frac{12}{5}$, vel $-\frac{12}{7}$ etc. Sit igitur primo $\frac{\varDelta-2a}{a}=-4$, vel $\varDelta=-2a$, unde nascitur haec fractio continua:

$$a+\cfrac{1}{-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\cfrac{1}{-7a+\text{ etc.}}}}}$$

quae manifesto a praecedente pendet. Si enim ponamus

$$-a+\cfrac{1}{-3a+\cfrac{1}{-5a+\text{ etc.}}}=s,$$

mutatis signis erit

$$-s=a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{ etc.}}}}$$

cuius valor cum iam sit inventus, iste casus nihil novi nobis offert.

22. Si sumatur

$$\frac{\varDelta-2a}{a}=-\frac{4}{3}, \text{ sive } \varDelta=\frac{2}{3}a,$$

inde sequens nascitur fractio continua:

1) Vide notam p. 175. H. D.

$$a+\cfrac{1}{\frac{5}{3}a+\cfrac{1}{\frac{7}{3}a+\cfrac{1}{\frac{9}{3}a+\text{etc.}}}}$$

sive statuendo $a=3\alpha$, erit fractio

$$3\alpha+\cfrac{1}{5\alpha+\cfrac{1}{7\alpha+\cfrac{1}{9\alpha+\text{etc.}}}}$$

quae est ipsa forma superior primo membro truncata. Idem usu venit, si sumatur $\frac{\varDelta-2a}{a}=-\frac{8}{3}$, sive $\varDelta=-\frac{2}{3}a$, unde oritur haec fractio continua, ponendo scilicet $a=3\alpha$:

$$3\alpha+\cfrac{1}{\alpha+\cfrac{1}{-\alpha+\cfrac{1}{-3\alpha+\cfrac{1}{-5\alpha+\text{etc.}}}}}$$

sicque semper ad principalem formam reducimur.

23. Casus autem integrabilitatis in genere hoc continentur exponente: $-\frac{4i}{2i\pm1}$. Posito igitur

$$\frac{\varDelta-2a}{a}=-\frac{4i}{2i\pm1}, \quad \text{fiet} \quad \varDelta=\frac{\pm2a}{2i\pm1},$$

unde posito $\frac{a}{2i\pm1}=\alpha$, erit $\varDelta=\pm2\alpha$, ergo ob $a=(2i\pm1)\alpha$ fractio continua erit

$$(2i\pm1)\alpha+\cfrac{1}{a\pm2\alpha+\cfrac{1}{a\pm4\alpha+\cfrac{1}{a\pm6\alpha+\text{etc.}}}}$$

ubi manifesto iterum omnes numeri impares occurrunt, ita ut fractio continua inde nata semper formari possit ex nostra principali

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

si vel aliquot membris truncetur, vel retro ad aliquot membra superne continuetur.

24. Cum igitur omnes casus integrabilitatis aequationis RICCATIANAE ad eandem fractionem continuam perducant, praeterea vero nulli adhuc alii casus evolvi potuerint; hinc manifesto sequitur, si indices fractionis continuae a, b, c, d, e, f etc. quamcunque aliam progressionem arithmeticam constituant, tum summam nullo plane modo assignari posse, quandoquidem ea pendet a casu irresolubili aequationis RICCATIANAE. Ita si proponatur haec fractio continua:

$$a+\cfrac{1}{2a+\cfrac{1}{3a+\cfrac{1}{4a+\text{etc.}}}}$$

ubi est $\varDelta = a$, summatio pendebit ab ista aequatione differentiali:

$$ady + yydt = \frac{dt}{t},$$

cuius resolutio cum per nullas quantitates transcendentes etiam nunc usu receptas expediri possit, valorem huius fractionis continuae frustra inter quantitates a circulo vel a logarithmis pendentes, vel adeo inter omnes quadraturas curvarum algebraicarum quaerimus; unde mirum non est, quod methodus in superiori dissertatione usitata pro talibus casibus omni successu caruerit.

25. Quoniam tamen in dissertatione superiore[1]) summam talium fractionum continuarum per binas formulas integrales expressam dedimus, illa ipsa

1) L. EULERI Commentatio 594, p. 198. Vide notam p. 174. H. D.

expressio etiam ad aequationem RICCATIANAM pro iisdem casibus accommodari poterit, id quod utique maximam attentionem meretur, cum nullo adhuc modo ista aequatio praeter casus integrabiles tractari potuerit. Quamobrem maxime operae erit pretium solutiones inter se comparare, quandoquidem hinc haud contemnendum subsidium, aequationem RICCATIANAM feliciori successu tractandi, expectari poterit.

26. In superiori autem dissertatione[1]) ostendi, si haec proposita fuerit fractio continua:

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{ etc.}}}}$$

eius valorem exprimi per hanc fractionem: $-\frac{A}{B}$, existente

$$A=\int\frac{dx}{x^{2+\frac{b}{m}}\cdot e^{\frac{1+xx}{mx}}}$$

et

$$B=\int\frac{dx}{x^{1+\frac{b}{m}}\cdot e^{\frac{1+xx}{mx}}},$$

siquidem haec duo integralia a termino $x=0$ usque ad terminum $x=\infty$ extendantur.

27. Comparemus nunc hanc fractionem continuam cum generali, quam hic tractavimus:

$$a+\cfrac{1}{a+\varDelta+\cfrac{1}{a+2\varDelta+\cfrac{1}{a+3\varDelta+\text{ etc.}}}}$$

cuius valor continetur in hac aequatione:

$$ady+yydt=t^{\frac{\varDelta-2a}{a}}dt,$$

1) L. EULERI Commentatio 594, p. 214. Vide notam p. 174. H. D.

integratione scilicet ita instituta, ut casu $t = 0$ fiat $y = \infty$; tum vero statuatur $t = 1$, unde valor ipsius y summam istius fractionis continuae exprimet.

28. Comparatione igitur instituta fiet

$$m = \varDelta \quad \text{et} \quad b = \varDelta - a,$$

quibus valoribus introductis formulae illae integrales erunt

$$A = \int \frac{dx}{x^{3-\frac{a}{\varDelta}} \cdot e^{\frac{1+xx}{\varDelta x}}}$$

et

$$B = \int \frac{dx}{x^{2-\frac{a}{\varDelta}} \cdot e^{\frac{1+xx}{\varDelta x}}},$$

integralibus iterum sumtis ab $x = 0$ ad $x = \infty$; quamobrem valor ipsius y, qui ex aequatione

$$ady + yydt = t^{\frac{\varDelta - 2a}{a}} dt$$

pro casu $t = 1$ resultat, aequalis erit isti fractioni: $-\frac{A}{B}$, sive erit

$$y = \frac{\int dx : x^{3-\frac{a}{\varDelta}} \cdot e^{\frac{1+xx}{\varDelta x}}}{\int dx : x^{2-\frac{a}{\varDelta}} \cdot e^{\frac{1+xx}{\varDelta x}}}.$$

Quanquam autem haec aequalitas tantum casus speciales, quibus ibi fit $t = 1$, hic vero $x = \infty$, spectat, tamen fortasse eiusmodi relatio inter binas variabiles t et x inveniri poterit, ut in genere quantitas y illi formulae aequetur.

29. Quemadmodum consideratio fractionis nostrae continuae nos ad resolutionem aequationis Riccatianae perduxit, ita vicissim datur methodus directa, qua ista aequatio per fractiones continuas resolvi potest. Quod quo facilius ostendatur, aequatio Riccatiana, quae vulgo hac forma proponi solet:

$$dy + ayydx = acx^{2m}dx,$$

in aliam formam ad praesens institutum magis accommodatam transfundatur, ponendo

$$y = x^m z \quad \text{et} \quad x^{m+1} = t;$$

tum enim, si loco $\frac{a}{m+1}$ scribatur b, prodit ista aequatio:

$$dz + \frac{mzdt}{(m+1)t} + bzzdt = bcdt.$$

30. Ponamus porro $\frac{m}{m+1} = -n$, ut sit $m = -\frac{n}{1+n}$ et nobis proposita sit ista aequatio:

$$dz - n\frac{zdt}{t} + bzzdt = bcdt,$$

quae, si adhibeamus hanc substitutionem:

$$z = \frac{n+1}{bt} + \frac{c}{v},$$

transmutatur in hanc formam:

$$dv - (n+2)\frac{vdt}{t} + bvvdt = bcdt,$$

quae a priore hoc tantum differt, ut hic numerus n binario maior sit factus; quare si porro faciamus

$$v = \frac{n+3}{bt} + \frac{c}{u},$$

prodibit haec aequatio:

$$du - (n+4)udt + buudt = bcdt,$$

unde patet, si prima aequatio resolutionem admittat casu $n = k$, tum etiam eius resolutionem in potestate fore casibus

$$n = k+2, \; n = k+4, \; n = k+6, \; \text{etc.}$$

et in genere casu $n = k + 2i$, denotante i numerum integrum quemcunque.

31. Quod si iam successive loco v et u et sequentium litterarum valores istos debitos substituamus, pro quantitate z sequens prodibit fractio continua:

$$z = \cfrac{n+1}{bt} + \cfrac{c}{\cfrac{n+3}{bt} + \cfrac{c}{\cfrac{n+5}{bt} + \text{ etc.}}}$$

quae ergo expressio exhibet valorem quantitatis z, qui ipsi convenit vi huius aequationis:

$$dz - n\frac{zdt}{t} + bzzdt = bcdt,$$

si scilicet ita integretur, ut posito $t = 0$ fiat $z = \infty$.

32. Liberemus hanc formam a fractionibus partialibus, et obtinebimus hanc formam:

$$\frac{1}{z} = \cfrac{bt}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\cfrac{bbctt}{n+7+\text{ etc.}}}}}$$

unde statim patet, posito $t = 0$ fore $\frac{1}{z} = 0$ ideoque $z = \infty$. Simili modo resolutionem ab aequatione transformata exordiri possumus, quae erat

$$dv - (n+2)\frac{vdt}{t} + bvvdt = bcdt,$$

quae in primam transformatur ponendo $v = \frac{bct}{-n-1+btz}$; prodibit enim

$$dz - \frac{nzdt}{t} + bzzdt = bcdt,$$

in qua aequatione numerus n binario redditus est minor; unde patet, si aequatio resolutionem admittat casu $n = k$, tum etiam resolutionem esse successuram casibus

$$n = k - 2, \; n = k - 4, \; n = k - 6$$

et in genere $n = k - 2i$. Quare cum resolutio nulla laboret difficultate casu $n = 0$, sumto $k = 0$ omnes casus resolutionem admittentes continebuntur in hac formula: $n = \pm\, 2i$ hincque pro forma consueta fiet

$$m = \frac{\mp 2i}{\pm 2i + 1} = -\frac{2i}{2i \pm 1},$$

quae continet casus notissimos integrabilitatis.

33. Ponamus $n + 2 = -\nu$, ut aequatio, a qua hic inchoamus, sit

$$dv + \frac{\nu v dt}{t} + bvvdt = bcdt,$$

quae ergo, posito

$$v = \frac{bct}{\nu + 1 + btz}$$

transmutatur in hanc formam:

$$dz + (\nu + 2)\frac{zdt}{t} + bzzdt = bcdt,$$

in qua nunc numerus ν binario augetur. Quare si ulterius ponamus

$$z = \frac{bct}{\nu + 3 + bty},$$

orietur haec aequatio:

$$dy + (\nu + 4)\frac{ydt}{t} + byydt = bcdt,$$

sicque ulterius progrediendo pervenietur ad $\nu + 5$, $\nu + 7$ etc.

34. Substituamus ergo istos valores in superiore aequatione, quae est

$$dv + \frac{\nu v dt}{t} + bvvdt = bcdt,$$

ac pro v prodibit sequens fractio continua:

$$v = \cfrac{bct}{\nu + 1 + \cfrac{bbctt}{\nu + 3 + \cfrac{bbctt}{\nu + 5 + \cfrac{bbctt}{\nu + 7 + \text{ etc.}}}}}$$

Haec igitur expressio locum habet, si aequatio differentialis ita integretur, ut posito $t = 0$ fiat $v = 0$.

35. Geminas igitur ex aequatione RICCATIANA elicuimus fractiones continuas, quas quo facilius inter se comparare queamus, loco v scribamus iterum n, ut habeamus has duas aequationes differentiales:

$$\text{I. } dz - n\frac{zdt}{t} + bzzdt = bcdt,$$

$$\text{II. } dv + n\frac{vdt}{t} + bvvdt = bcdt,$$

atque ex priore nascetur ista fractio continua:

$$\frac{1}{z} = \cfrac{bt}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\text{ etc.}}}}$$

ex altera vero oritur

$$v = \cfrac{bct}{n+1+\cfrac{bbctt}{n+3+\cfrac{bbctt}{n+5+\text{ etc.}}}}$$

quae ergo fractiones prorsus inter se conveniunt, cum hinc fiat $v = \frac{c}{z}$, quippe quo modo altera aequatio in alteram actu convertitur, ita ut hae duae formae pro unica sint habendae.

36. Ista resolutio aequationis RICCATIANAE in fractionem continuam eo magis est attentione digna, quod haec aequatio nullo adhuc modo in seriem infinitam regularem resolvi potuerit. Quas enim series olim pro eius resolutione exhibui[1]), ita sunt comparatae, ut una series infinita per aliam divisa resolutionem aequationis RICCATIANAE suppeditet; hic autem de unica serie simplici sermo instituitur. Hinc igitur nascitur quaestio, num forte non etiam aliae aequationes differentiales dentur, quarum resolutionem pariter per fractiones continuas expedire liceat.

1) Vide *Institutionum calculi integralis* vol. II, § 940. Cf. quoque Commentationes 269, 284 (indicis *Enestroemiani*): *De integratione aequationum differentialium.* Novi Comment. acad. sc. Petrop. 8, 1763, p. 44, *De resolutione aequationis* $dy + ayydx = bx^m dx$. Novi Comment. acad. sc. Petrop. 9, 1764, p. 154. *LEONHARDI EULERI Opera omnia*, series I, vol. 12, p. 157, vol. 22, p. 334 et 403. H. D.

SPECIMEN SINGULARE ANALYSEOS INFINITORUM INDETERMINATAE

Conventui exhibita die 18. Martii 1776
Commentatio 622 indicis ENESTROEMIANI
Nova acta Academiae scientiarum Petropolitanae 3 (1785), 1788, p. 47—56
Summarium ibidem p. 168—170

SUMMARIUM

Parmi le grand nombre de découvertes dont les sciences mathématiques sont redevables à l'Immortel EULER, la méthode de résoudre, par une voye directe, des Problèmes de l'Analyse infinitésimale analogues à ceux que la méthode de DIOPHANTE enseigne à résoudre dans l'Algèbre, ne tient pas la dernière place. Les Géomètres savent quelle grande sensation a faite, en son tems, le Problème proposé et résolu par HERMANN, de trouver une courbe algébrique non rectifiable, mais dont la rectification dépend de la quadrature d'une courbe donnée, douée de tant qu'on voudrait d'arcs rectifiables; ils savent que feu M. EULER a été le premier à résoudre ce Problème d'une manière directe dans un mémoier intitulé: *De methodo Diophanteae analoga in Analysi Infinitorum*, qu'on trouve dans le Tome cinquième des nouveaux Commentaires de notre Académie, et dans lequel il a jetté les premiers fondemens de ce nouveau Calcul et exposé la solution de quantité de Problèmes relatifs à cette méthode et résolus directement par le moyen de ses nouveaux principes.

Cependant le Problème le plus général qu'on peut résoudre par le moyen de cette nouvelle méthode, c'est de trouver une telle relation algébrique entre les deux quantités variables x et y, que toutes ces formules intégrales: $\int P\partial y$, $\int Q\partial y$, $\int R\partial y$ etc. obtiennent des valeurs algébriques, et qu'une ou deux d'entr'elles renferment des quadratures données, les lettres P, Q, R marquant des fonctions quelconques données de x. Mais quoique ce cas paroisse être d'une grande généralité, il est restreint par la condition que la variable y ne doit avoir, dans ces formules, qu'une seule dimension; et toutes les fois que le Problème indéterminé qu'on traite n'est pas réductible à de pareilles formules, la méthode de

M. Euler, exposée dans le mémoire cité, est insuffisante; et il avoue ici lui même qu'il ne voit pas comment tenter seulement la solution des cas qui ne sont pas contenus dans les formules mentionnées. Il en allègue pour exemple les deux formules si simples $\int\frac{y\partial x}{x}$ et $\int\frac{\partial x}{yx}$, dont on ne voit pas comment trouver l'intégrale, à moins qu'on ne prenne pour y une puissance quelconque de x, quoi qu'il y ait lieu de présumer que ce ne soit pas la seule solution possible.

La perfection de ce genre de calcul paroît donc promettre une riche recolte de vérités nouvelles et de nouveaux moyens de résoudre plusieurs Problèmes qui jusqu'ici se sont refusés à tous les efforts des Géomètres. C'est de là que dépend, par exemple, la démonstration complette des deux Théorèmes, dont feu M. Euler avoit tâché de montrer la vérité dans ses Opuscules Analytiques T. II, page 82 et suivantes, savoir: 1°) Qu'à l'exception du cercle il n'y ait point de courbe algébrique dont chaque arc puisse être exprimé par un arc de cercle; et 2°) Qu'il n'y ait point de courbe algébrique dont les arcs puissent être exprimés simplement par des logarithmes. De même la recherche des lignes rectifiables tirées sur une surface courbe donnée, soit convexe ou concave, est sujette à de grandes difficultés qu'on ne surmontera apparemment jamais sans le secours de ce nouveau genre de Calcul mieux perfectionné, c'est pourquoi l'Auteur invite tous les Géomètres à s'appliquer à cette partie de l'Analyse.

La solution du Problème qui fait le sujet du présent mémoire est très propre à repandre quelque jour sur ce genre ténébreux de calcul, quoi qu'elle soit extrèmement indirecte. Il s'agit de trouver une telle relation entre les variables q et z, que la formule $\int q\partial z$ devienne algébrique, et que la formule

$$\int\frac{\partial z\sqrt{(qq-1)}}{z}$$

exprime un arc de cercle. Comme la dernière condition est la plus difficile à remplir, l'Auteur commence par là, en mettant

$$\int\frac{\partial z\sqrt{(qq-1)}}{z} = \text{A tang.}\,\frac{x}{y},$$

d'où, en prenant les différentielles et mettant $xx+yy=zz$, il déduit

$$\sqrt{(qq-1)} = \frac{y\partial x - x\partial y}{x\partial x + y\partial y} \quad \text{et} \quad q = \frac{z\sqrt{(1+pp)}}{x+py},$$

où $p=\frac{\partial y}{\partial x}$. De cette manière les deux quantités variables q et z sont déterminées par deux autres x et y, et la dernière condition est remplie. Pour satisfaire à l'autre condition, comme

$$q\partial z = \partial x\sqrt{(1+pp)} \quad \text{et} \quad \partial y = p\partial x,$$

tout se reduit à rendre algébriques ces deux formules:

$$y=\int p\partial x \quad \text{et} \quad \int q\partial z = \int\partial x\sqrt{(1+pp)}.$$

Or

$$\int p\partial x = px - \int x\partial p\ ;$$

$$\int \partial x \sqrt{(1+pp)} = x\sqrt{(1+pp)} - \int \frac{xp\partial p}{\sqrt{(1+pp)}},$$

où les deux dernières formules sont contenues dans celles que M. EULER a enseigné à rendre algébriques dans le Problème général dont nous avons fait mention. Cependant il donne ici une double solution de ces deux formules, l'une par les irrationnelles, l'autre par des expressions dégagées de l'irrationalité.

Quoique cette méthode de résoudre le Problème soit très indirecte, et due uniquement à la circonstance que la solution en étoit déjà connue d'avance: elle peut neanmoins donner aux Géomètres l'occasion de perfectionner ce nouveau genre de calcul si propre à reculer les bornes de l'Analyse et c'est dans cette espérance principalement que M. EULER a communiqué cette solution.

1. Iam ante complures annos novum prorsus Calculi genus adumbravi, cui *Analyseos Infinitorum indeterminatae*[1]) nomen inposueram, quoniam ad Analysin Infinitorum ordinariam eodem modo refertur, quo Analysis Diophantea ad Algebram communem. Indoles scilicet huius Calculi in eo consistit, ut eiusmodi relatio inter binas variabiles investigetur, unde una pluresve formulae integrales nanciscantur valores sive algebraicos, sive datas quadraturas involventes. Veluti si talis definiri debeat relatio inter binas variabiles x et y, ut ista formula integralis: $\int\sqrt{(\partial x^2 + \partial y^2)}$ algebraicum valorem adipiscatur, vel etiam datas quantitates transcendentes involvat. Hinc enim evidens est curvas algebraicas obtineri, quae sint vel rectificabiles, vel quarum rectificatio a datis quadraturis pendeat; atque hinc Problema illud HERMANNIANUM celeberrimum methodo directa solutum dedi[2]), quo requirebantur curvae algebraicae non rectificabiles, sed quarum rectificatio datas quadraturas involveret, in quibus tamen nihilominus vel unus, vel duo, vel adeo quotquis voluerit arcus assignari possent absolute rectificabiles, postquam ipse HERMANNUS et BERNOULLII[3]) methodo maxime indirecta ad eius solutionem pervenissent.

1) L. EULERI Commentationes 23, 48, 245: *De curvis rectificabilibus algebraicis utque trajectoriis reciprocis algebraicis; Investigatio binarum curvarum, quarum arcus eidem abscissae respondentes summam algebraicam constituant.* Comment. acad. sc. Petrop. 5, 1738, p. 169; 8, 1741, p. 23; *De methodo Diophanteae analoga in analysi infinitorum.* Novi Comment. acad. sc. Petrop. 1760, p. 84. *LEONHARDI EULERI Opera omnia*, series I, vol. 27, vol. 22, p. 76 et 237. Vide quoque Commentationes 650, 779, 784, 856 huius voluminis. H. D.

2) Commentatio 245, § 70. Vide notam 1). H. D.

3) IAC. HERMANN (1678—1733), *Solutio propria duorum problematum geometricorum in Actis Erudit.* 1719 *Mens. Aug. a se propositorum*, Acta erud. 1723, p. 171. IOH. BERNOULLI (1667—1748), *Methodus inveniendi curvas algebraicas indefinite non quadrabiles habentem tamen numerum determinatum spatiorum absolute quadrabilium.* Acta erud. Suppl. t. VIII, 1724, p. 380; *Methodus commoda et naturalis reducendi quadraturas transcendentes cuiusvis gradus ad longitudinem curvarum algebraicarum.* Acta erud. 1724, p. 356. Opera omnia t. II p. 315 et 582. H. D.

Praeterea vero etiam tum temporis plura alia huius generis Problemata non parum curiosa ope methodi, quam ibi exposui, felici successu expedivi.

2. Methodus autem mea huiusmodi Problemata solvendi ita est comparata, ut eius beneficio sequens Problema generale pertractari possit:

Si P, Q, R, S etc. fuerint functiones quaecunque datae variabilis x, semper eiusmodi relatio algebraica inter binas variabiles x et y assignari potest, ut omnes istae formulae integrales: $\int P\partial y$, $\int Q\partial y$, $\int R\partial y$ etc., quotcunque fuerint, algebraicos sortiantur valores[1]*). Quin etiam effici potest ut una earum, vel etiam duae, datas quadraturas involvant.*

Quamvis autem iste casus latissime patere videatur, tamen hac conditione maxime restringitur, quod in istis formulis altera variabilis y unicam tantum obtineat dimensionem. Si enim diversae dimensiones occurrerent, neutiquam adhuc perspicere possum, quomodo resolutio suscipi deberet.

3. Quoties igitur eiusmodi quaestiones proponuntur, quas ad huiusmodi formulas revocare non licet, fateri cogor, me nullo adhuc modo perspicere posse, quibusnam artificiis solutionem saltem tentari conveniat, id quod exemplo simplicissimo declarasse sufficiet. Veluti si hae duae formulae: $\int\frac{y\partial x}{x}$ et $\int\frac{\partial x}{yx}$, ambae reddi debeant integrabiles, aliam solutionem exhiberi posse non video, nisi quae sponte se offert, dum pro y potestas quaecunque ipsius x assumitur. Vix autem asseverare ausim, nullam aliam solutionem locum habere posse. Ex quo intelligere licet, quantopere adhuc istud novum calculi genus nobis sit absconditum, et omnia quae adhuc sunt praestita vix tanquam prima eius elementa spectari posse; unde maxime esset optandum, ut sagacissima ingenia omnes vires intenderent ad istam Analyseos partem uberius excolendam.

4. Ad hoc etiam Calculi genus referri debent bina illa Theoremata, quae non ita pridem in medium afferre sum ausus, quorum priore asseveravi[2]), praeter circulum nullam aliam dari curvam algebraicam, cuius singuli arcus per

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1) Vide Commentationes 650, 779 huius voluminis. H. D.

2) Vide L. EULERI Commentationes 590, 783: *Theoremata quaedam analytica, quorum demonstratio adhuc desideratur.* Opuscula analytica 2, 1785, p. 76. *De curvis algebraicis, quarum omnes arcus per arcus circulares metiri liceat.* Memoires acad. sc. Petersb. 11, 1830, p. 114. *LEONHARDI EULERI Opera omnia* series I, vol. 21, p. 78 et 262. H. D.

arcus circulares exhiberi queant, sive nullam aliam inter x et y relationem algebraicam assignari posse, ut fieret

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial z}{\sqrt{(1-zz)}}.$$

Altero autem Theoremate affirmavi, nullam plane dari curvam algebraicam, cuius singuli arcus per simplices logarithmos exprimi queant, sive ut fieri possit

$$\sqrt{(\partial x^2 + \partial y^2)} = \frac{\partial z}{z}.$$

Pluribus quidem rationibus veritatem horum Theorematum corroborare sum annisus, ita ut nullum amplius dubium superesse posse videatur; interim tamen plenam eius demonstrationem vix ante exspectare licebit, quam novum hoc Calculi genus uberius fuerit elaboratum.

5. Imprimis autem ad istum Calculum pertinent quaestiones iam passim tractatae de lineis rectificabilibus in data superficie sive convexa sive concava ducendis[1]), quarum solutio semper multo maiorem huius novi calculi perfectionem requirere videtur. Postquam enim multum desudassem, ut in superficie sphaerica lineam rectificabilem investigarem, nullam aliam reperire potui, praeter eam, quae Geometris iam pridem innotuit, quae scilicet describitur, dum circulus sphaerae maximus super minore provolvitur, et cuius inventio casui potius fortuito quam certae methodo accepta est referenda; unde vix dubitaverim asseverare, praeter istam curvam in superficie sphaerica nullam aliam dari, quae esset rectificabilis. Quin etiam in superficiebus cylindricis et conicis nullae aliae lineae rectificabiles mihi quidem exhiberi posse videntur, praeter eas, quae ipsae sunt rectae.

6. Nuper vero se mihi alia huius generis quaestio obtulit, cuius quidem solutio iam aliunde mihi erat cognita; interim tamen eam ita comparatam deprehendi, ut nullam plane viam directam, ad eius solutionem pertingendi, perspicere potuerim, nisi ipsa solutio iam aliunde innotuisset. Hinc scilicet

1) Vide L. Euleri Commentationes 408, 574, 623: *De curva rectificabili in superficie sphaerica.* Novi Comment. acad. sc. Petrop. **15**, 1771, p. 196; *De curvis rectificabilibus in superficie coni recti ducendis.* Acta acad. sc. Petrop. 1781: **5** I, 1784, p. 60; *De lineis rectificabilibus in superficie sphaeroidica quacumque geometrice ducendis.* Nova acta acad. sc. Petrop. **3**, 1788, p. 96. *Leonhardi Euleri Opera omnia*, series I, vol. **27** et **28**. H. D.

methodum maxime obliquam et indirectam derivavi, quae demum per plures ambages ad scopum perduxerat; quamobrem plurimum lucis in hoc obscuro calculi genere affulgere posse confido, si problema istud cum mea solutione, quantumvis obliqua, Geometris proposuero.

PROBLEMA

Investigare relationem inter binas quantitates variabiles q et z, ut primo haec formula: $\int q\,dz$ fiat algebraica, tum vero ista: $\int\frac{\partial z\sqrt{(qq-1)}}{z}$ arcum circularem exprimat.

SOLUTIO

7. Hic manifesto summa difficultas in posteriori formula deprehenditur, quam ad arcum circularem revocari oportet; ubi quidem in transitu notari meretur, si ista formula etiam algebraica reddi deberet, uti prior, ne ullam plane viam me perspicere posse hoc negotium conficiendi. Ex quo conditio, quod ista formula ad arcum circularem reduci debeat, multo difficilior videbatur; interim tamen haec ipsa conditio viam nobis apperiet ad scopum propositum perveniendi, id quod ex sequentibus operationibus patebit.

8. Primo igitur formulam

$$\int\frac{\partial z\sqrt{(qq-1)}}{z}$$

arcui circuli, cuius tangens sit $\frac{x}{y}$, aequalem statuamus, ubi data opera duas novas variabiles x et y in calculum introducimus, quo deinceps ambae nostrae propositae q et z per eas commodius exprimi queant. Sumtis igitur differentialibus consequemur hanc aequationem:

$$\frac{\partial z\sqrt{(qq-1)}}{z}=\frac{y\partial x-x\partial y}{xx+yy},$$

et nunc faciamus $zz=xx+yy$, id quod utique sine ulla quaestionis restrictione fieri licet, propterea quod ambae litterae x et y prorsus a nostro arbitrio pendent. Posito autem

$$zz=xx+yy,$$

superest, ut fiat

$$z\partial z\sqrt{(qq-1)}=y\partial x-x\partial y,$$

unde, cum sit

$$z\partial z=x\partial x+y\partial y,$$

deducimur ad hanc determinationem:

$$\sqrt{(qq-1)}=\frac{y\partial x-x\partial y}{x\partial x+y\partial y}.$$

9. Ut iam hanc formulam a differentialibus liberemus, statuamus $\partial y=p\partial x$, ubi manifestum est formulam integralem $\int p\partial x$ absolute integrabilem seu algebraicam reddi debere. Hinc igitur habebimus

$$\sqrt{(qq-1)}=\frac{y-px}{x+py},$$

ubi commode usu venit, ut sumtis quadratis fiat

$$qq=\frac{(1+pp)(xx+yy)}{(x+py)^2};$$

quare ob $xx+yy=zz$, erit radice extracta

$$q=\frac{z\sqrt{(1+pp)}}{x+py},$$

ita ut nunc ambae variabiles q et z propositae satis concinne per binas novas variabiles x et y expressae prodierint, atque posteriori conditioni, qua formula $\int\frac{\partial z\sqrt{(qq-1)}}{z}$ arcum circuli exprimere debet, iam perfecte sit satisfactum, idque tam generaliter, ut nulla limitatio sit introducta, quandoquidem in calculo adhuc duae variabiles x et y remanserunt, nullo modo a se invicem pendentes.

10. Quoniam igitur posteriori conditioni Problematis est satisfactum, nihil aliud superest, nisi ut prior conditio, qua formula $\int q\partial z$ ad quantitatem algebraicam est revocanda, adimpleatur. Quod si vero loco q valorem inventum substituamus, reperiemus

$$q\partial z=\frac{z\partial z\sqrt{(1+pp)}}{x+py};$$

quare cum sit

$$z\partial z = x\partial x + y\partial y = \partial x(x + py),$$

quasi praeter exspectationem deducimur ad istam formulam simplicissimam:

$$q\partial z = \partial x\sqrt{(1 + pp)},$$

quandoquidem hinc denominator quasi casu fortuito est sublatus, ex quo tota quaestio huc est reducta, ut ista formula integralis:

$$\int \partial x \sqrt{(1 + pp)}$$

ad quantitatem algebraicam revocetur, simul vero etiam, uti iam ante observavimus, haec formula $\int p\partial x$ evadat algebraica, quibus duabus conditionibus cum fuerit satisfactum, problema nostrum in omni extensione simul erit resolutum; tum enim primo habebimus $y = \int p\partial x$, hincque porro

$$z = \sqrt{(xx + yy)} \quad \text{et} \quad q = \frac{z\sqrt{(1 + pp)}}{x + py}.$$

His autem valoribus ambae conditiones praescriptae ita adimplentur, ut sit

$$\int q\partial z = \int \partial x \sqrt{(1 + pp)},$$

quae per hypothesin est quantitas algebraica; pro altera autem conditione fit

$$\int \frac{\partial z \sqrt{(qq - 1)}}{z} = \text{A tang.}\, \frac{x}{y},$$

ideoque arcui circuli aequalis, uti requirebatur.

11. Quaeri igitur debet eiusmodi relatio inter binas variabiles p et x, ut ambae istae formulae:

$$\int p\partial x \quad \text{et} \quad \int \partial x \sqrt{(1 + pp)}$$

evadant algebraicae. Cum igitur sit

$$\int p\partial x = px - \int x\partial p$$

et

$$\int \partial x \sqrt{(1+pp)} = x\sqrt{(1+pp)} - \int \frac{xp\partial p}{\sqrt{(1+pp)}},$$

quoniam hae duae novae formulae in supra § 2 memoratis continentur, per methodum olim expositam[1]) statuamus primo

$$\int x\partial p = t, \quad \text{ut sit} \quad x = \frac{\partial t}{\partial p},$$

atque altera formula abibit in hanc:

$$\int \frac{p\partial t}{\sqrt{(1+pp)}},$$

quae si statuatur $= u$, hinc fiet

$$\frac{p}{\sqrt{(1+pp)}} = \frac{\partial u}{\partial t};$$

ubi iam pro u functionem quamcunque algebraicam ipsius t assumere licet. Quare si ponamus

$$\partial u = v\partial t,$$

erit

$$\frac{p}{\sqrt{(1+pp)}} = v,$$

hincque

$$p = \frac{v}{\sqrt{(1-vv)}} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{1}{\sqrt{(1-vv)}}.$$

12. Tota ergo nostra solutio ita se habebit. Sumta functione quacunque variabilis t, quae vocetur $= u$, unde fiat $\partial u = v\partial t$, ita ut etiam v sit functio ipsius t, hinc primo erit

$$p = \frac{v}{\sqrt{(1-vv)}} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{1}{\sqrt{(1-vv)}},$$

quo valore invento capiatur

$$x = \frac{\partial t}{\partial p},$$

ita ut iam p et x per solam variabilem t sint expressae. Tum vero erit

1) L. EULERI Commentatio 245. Cf. § 70. Vide notam 1), p. 197. H. D.

$$\int p\partial x = \frac{v\partial t}{\partial p\sqrt{(1-vv)}} - t = y.$$

Deinde simili modo erit

$$\int \partial x\sqrt{(1+pp)} = \frac{\partial t}{\partial p\sqrt{(1-vv)}} - u = \int q\partial z.$$

Denique ex his ipsae quantitates in Problemate quaesitae ita per novam variabilem t exprimentur, ut sit

$$z = \sqrt{(xx+yy)},$$

et hinc

$$q = \frac{z\sqrt{(1+pp)}}{x+py},$$

atque ex his valoribus Problemati ita satisfiet, ut sit

$$\int q\partial z = \frac{x}{\sqrt{(1-vv)}} - u \quad \text{et} \quad \int \frac{\partial z\sqrt{(qq-1)}}{z} = \text{A tang.}\ \frac{x}{y}.$$

13. Quo haec clarius intelligantur, exemplum evolvamus sumendo $u = at^n$, unde fit

$$v = nat^{n-1} \quad \text{et} \quad \sqrt{(1-vv)} = \sqrt{(1-nnaat^{2n-2})},$$

unde colligitur

$$p = \frac{nat^{n-1}}{\sqrt{(1-nnaat^{2n-2})}} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{1}{\sqrt{(1-nnaat^{2n-2})}}.$$

Cum igitur hinc sit

$$\partial p = \frac{n(n-1)at^{n-2}\partial t}{(1-nnaat^{2n-2})^{\frac{3}{2}}},$$

erit

$$x = \frac{(1-nnaat^{2n-2})^{\frac{3}{2}}}{n(n-1)at^{n-2}}$$

et

$$y = \frac{t(2-n-nnaat^{2n-2})}{n-1}.$$

Quoniam vero hae formulae iam nimis fiunt intricatae, sumamus

$$a = 1 \quad \text{et} \quad n = 2, \quad \text{ut sit} \quad u = tt \quad \text{et} \quad v = 2t,$$

ex quibus porro colliguntur valores

$$p = \frac{2t}{\sqrt{(1 - 4tt)}} \quad \text{et} \quad \sqrt{(1 + pp)} = \frac{1}{\sqrt{(1 - 4tt)}},$$

$$x = \tfrac{1}{2}(1 - 4tt)^{\frac{3}{2}}, \quad y = -4t^3,$$

hincque

$$z = \sqrt{(\tfrac{1}{4} - 3tt + 12t^4)},$$

ac denique

$$q = \frac{\sqrt{(\frac{1}{4} - 3tt + 12t^4)}}{\frac{1}{2} - 4tt}.$$

Problemati autem iam ita satisfiet, ut sit

$$\int q\partial z = \tfrac{1}{2} - 3tt$$

et

$$\int \frac{\partial z \sqrt{(qq - 1)}}{z} = \text{A tang.} - \frac{(1 - 4tt)^{\frac{3}{2}}}{8t^3}.$$

ALIA SOLUTIO

PER FORMULAS RATIONALES PROCEDENS

14. Ut formulas radicales evitemus, ponamus statim

$$p = \frac{rr - 1}{2r}, \quad \text{ut fiat} \quad \sqrt{(1 + pp)} = \frac{rr + 1}{2r}.$$

Nunc igitur has duas formulas:

$$\int p\partial x = \int \frac{(rr - 1)\partial x}{2r}$$

et

$$\int \partial x \sqrt{(1 + pp)} = \int \frac{(rr + 1)\partial x}{2r},$$

integrabiles reddi oportet, id quod praestabitur faciendo has duas formulas: $\int r\partial x$ et $\int \frac{\partial x}{r}$ integrabiles; tum enim fiet

$$\int p\partial x = \tfrac{1}{2}\int r\partial x - \tfrac{1}{2}\int\frac{\partial x}{r}$$

et

$$\int \partial x \sqrt{(1+pp)} = \tfrac{1}{2}\int r\partial x + \tfrac{1}{2}\int\frac{\partial x}{r}.$$

Sicque quaeritur eiusmodi relatio inter r et x, ut tantum hae duae formulae evadant integrabiles.

15. Hic igitur reductione supra adhibita utamur, et cum sit

$$\int r\partial x = rx - \int x\partial r \quad \text{et} \quad \int\frac{\partial x}{r} = \frac{x}{r} + \int\frac{x\partial r}{rr},$$

ponamus

$$\int x\partial r = t, \quad \text{ut fiat} \quad x = \frac{\partial t}{\partial r};$$

tum vero altera formula erit

$$\int\frac{x\partial r}{rr} = \int\frac{\partial t}{rr},$$

cuius valorem si vellemus statuere $= u$, quantitas r signum radicale involveret, quod ut evitemus, utamur eadem reductione:

$$\int\frac{\partial t}{rr} = \frac{t}{rr} + 2\int\frac{t\partial r}{r^3},$$

ac iam statuamus

$$\int\frac{t\partial r}{r^3} = s, \quad \text{fietque} \quad t = \frac{r^3\partial s}{\partial r}.$$

Hic iam pro s functionem quamcunque ipsius r accipere licebit, unde fiat $\partial s = s'\partial r$, sicque habebimus

$$t = r^3 s',$$

hincque regrediendo

$$\int\frac{\partial t}{rr} = \int\frac{x\partial r}{rr} = rs' + 2s,$$

tum vero, posito $\partial s' = s''\partial r$, erit

$$x = 3rrs' + r^3 s''.$$

16. His valoribus inventis, cum sit

$$\int r\partial x = 2r^3 s' + r^4 s''$$

et

$$\int \frac{\partial x}{r} = 2s + 4rs' + rrs'',$$

ex his duobus valoribus colligimus sequentes:

$$\int p\partial x = -s + (r^3 - 2r)s' + \tfrac{1}{2}(r^4 - rr)s'',$$
$$\int \partial x \sqrt{(1 + pp)} = s + (r^3 + 2r)s' + \tfrac{1}{2}(r^4 + rr)s'',$$

ex quibus porro deducuntur reliqui:

$$\int p\partial x,\ z = \sqrt{(xx + yy)},\ q = \frac{z\sqrt{(1 + pp)}}{x + py},$$

hocque pacto tota solutio formulis rationalibus absolvetur.

17. Ceterum non dubito, quin contemplatio huius Problematis et Solutionis utcunque obliquae, quae casu tantum successisse videatur, Geometris occasionem suppeditare queat, hoc novum Calculi genus, cuius vix prima elementa nobis etiamnunc sunt cognita, ulterius rimandi atque ad maiorem perfectionis gradum evehendi, quandoquidem hinc maxima incrementa ad universam Analysin redundare sunt existimanda.

DE FORMULIS DIFFERENTIALIBUS QUAE PER DUAS PLURESVE QUANTITATES DATAS MULTIPLICATAE FIANT INTEGRABILES

Conventui exhibita die 1. Iulii 1776

Commentatio 650 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae 7 (1789), 1793, p. 3—21

Summarium ibidem p. 35—36

SUMMARIUM

Nous avons déjà eu l'occasion de parler à différentes reprises des recherches que feu M. EULER a faites sur les courbes algébriques dont les arcs peuvent être exprimés par des formules intégrales données. Dans les extraits qui se trouvent à la tête du cinquième volume des Nouveaux Actes de l'Académie, nous avons fait mention de deux Mémoires, où cet illustre Géomètre a trouvé une infinité de courbes algébriques dont les arcs peuvent être mésurés par des arcs paraboliques ou elliptiques, et dans l'Histoire de l'année suivante on trouve la notice d'un Mémoire, où cet Auteur cherchoit toutes les courbes dont les arcs sont exprimés par la formule générale

$$\int \frac{v^{m-1}dv}{\sqrt{(1-v^{2n})}},$$

et par cette autre beaucoup plus générale:

$$\int \frac{v^{m-1}dv}{\sqrt{(1-v^{2n})}}[a+bv^{2n}+cv^{4n}+dv^{6n}+ \text{ etc.}],$$

v étant une fonction des ordonnées, qu'il s'agit de déterminer, de manière que ces formules expriment l'arc de la courbe indéfini. Mais malgré tous ces heureux efforts de répandre du jour sur cette nouvelle partie de l'Analyse, il faut convenir qu'on ignore encore jusqu'aux premiers principes l'art de résoudre généralement ces sortes de questions; et ce n'est qu'à force de multiplier de pareilles recherches qu'on peut espérer d'approfondir cette matière digne de l'attention de tous les Géomètres.

Le présent Mémoire nous paroît fort propre à éclaircir en plusieurs points cette nouvelle branche de calcul, que M. EULER a été le premier, et presque le seul, à traiter sous le nom d'Analyse indéterminée des Infinis: il est une suite des recherches antérieures de l'Auteur sur le même sujet. Car comme le Problème général de trouver des courbes algébriques dont l'élément indéfini puisse être exprimé par une formule différentielle préscrite ∂s, se reduit à trouver un angle φ tel que les formules

$$\partial s \sin.\varphi \quad \text{et} \quad \partial s \cos.\varphi$$

deviennent intégrables, et que cette question ne peut même être tentée en général, M. EULER traite ici le Problème inverse, en cherchant toutes les formules différentielles ∂W qui, multipliées par deux quantités proposées quelconques, deviennent intégrables.

Après avoir résolu ce Problème en général, et de deux manières différentes, l'Auteur fait l'application de ses formules générales à quelques cas particuliers qui sont d'un grand usage dans les recherches sur la nature des lignes courbes, relativement à leur rectification.

M. EULER termine son Mémoire par la résolution d'un Problème encore plus général, dans lequel il cherche une formule différentielle telle que si on la multiplie par trois quantités variables proposées, chaque produit se prête à l'intégration.

1. Iam saepius eiusmodi quaestiones tractavi, quibus curvae algebraicae requiruntur, quarum longitudo per datam formulam integralem exprimatur[1]). Ita nuper[2]) infinitas curvas algebraicas mihi quidem assignare licuit, quarum longitudo sive per arcus Parabolicos, sive Ellipticos mensurari queat; tum vero etiam plures alias formulas, quibus longitudo curvae exprimatur, satis felici successu sum perscrutatus[3]). Interim tamen ex his omnibus concludi debet, eam Analyseos partem, ad quam huiusmodi quaestiones sunt referendae, minime adhuc esse satis excultam, atque adeo etiamnunc quasi prima principia latere, unde huiusmodi quaestionum solutionem peti oporteat. Plurimum igitur ad fines Analyseos promovendos conferre putandum est, si hoc argumentum Geometrae omni cura ulterius prosequi dignabuntur.

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1) Vide L. EULERI Commentationes 633, 638, 639, 645: *De binis curvis algebraicis inveniendis, quarum arcus indefinite inter se sint aequales; De innumeris curvis algebraicis, quarum longitudinem per arcus parabolicos metiri licet; De innumeris curvis algebraicis, quarum longitudinem per arcus ellipticos metiri licet; De curvis algebraicis, quarum longitudo exprimitur hac formula integrali* $\int \frac{v^{m-1}dv}{\sqrt{(1-v^{2n})}}$ Nova acta acad. sc. Petrop. 4 (1786), 1789, p. 96; 5 (1787), 1789, p. 59; 5 (1787), 1789, p. 71; 6 (1788), 1790, p. 36. Vide quoque Commentationes 780, 781, 782, 817: *De infinitis curvis algebraicis, quarum longitudo indefinita arcui elliptico aequatur; De infinitis curvis algebraicis, quarum longitudo arcui parabolico aequatur; De binis curvis algebraicis eadem rectificatione gaudentibus.* Memoires acad. sc. Petersb. 11, 1830, p. 95, p. 100, p. 102; *De lineis curvis, quarum rectificatio per datam quadraturam mensuratur.* Opera postuma 1, Petrop. 1862, p. 439. *LEONHARDI EULERI Opera Omnia*, series I, vol. 21. Vide porro notas 1) p. 197, 2) p. 198, 1) p. 199 huius voluminis. H. D.

2) C. Nov. Act. Acad. Tom. V. pro Anno 1787.

3) C. Nov. Act. Acad. Tom. VI. pro Anno 1788.

2. Quando autem quaestio proponitur de curvis Algebraicis inveniendis, quarum elementum indefinitum formula differentiali quadam praescripta ∂s exprimatur, totum negotium eo reducitur, ut angulus quispiam φ investigetur, ex quo hae duae formulae differentiales $\partial s \sin.\varphi$ et $\partial s \cos.\varphi$ integrabiles evadant. Quae investigatio cum in genere ne suscipi quidem queat, quaestionem inversam accuratius tractasse iuvabit, qua omnes eae formulae differentiales exquiruntur, quae tam per $\sin.\varphi$ quam per $\cos.\varphi$ multiplicatae reddantur integrabiles, cuius resolutio cum nulla amplius laboret difficultate, eam in latiori sensu acceptam evolvamus, quo loco formularum $\sin.\varphi$ et $\cos.\varphi$ aliae quantitates quaecunque proponuntur. Quin etiam istam quaestionem ad tres pluresve huiusmodi quantitates extendamus. Quanquam autem methodum huiusmodi problemata solvendi iam ante complures annos adumbravi, qua nova quaedam pars Analyseos Infinitorum, quam *indeterminatam*[1]) appellare liceat, constitui est censenda, tamen quoniam hoc argumentum tum nimis generaliter est tractatum, nunc operae pretium erit id maiori cura propius ad praesens institutum accommodare.

PROBLEMA 1

Investigare omnes formulas differentiales, quae per datas duas quantitates propositas multiplicatae reddantur integrabiles[2]).

SOLUTIO

3. Designemus formulam differentialem quaesitam charactere ∂W, sintque p et q bini illi multiplicatores dati, quibus haec formula integrabilis reddi debeat; ita ut hae duae formulae integrales: $\int p\,\partial W$ et $\int q\,\partial W$ evadant quantitates algebraicas. Denotent igitur P et Q istas quantitates algebraicas, ut sit $\int p\,\partial W = P$ et $\int q\,\partial W = Q$, atque hinc duplici modo sponte elicitur

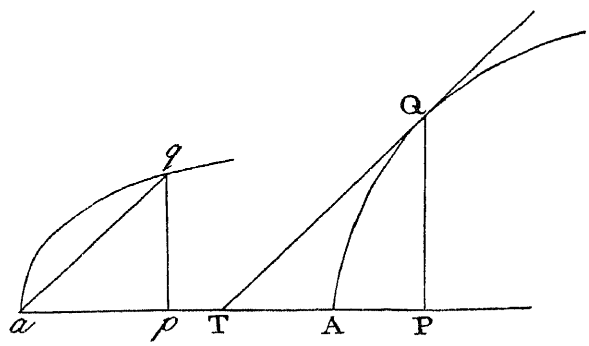
$$\partial W = \frac{\partial P}{p} \quad \text{et} \quad \partial W = \frac{\partial Q}{q}.$$

Nunc igitur quaestio perducta est ad duas quantitates algebraicas P et Q investigandas, quarum differentialia inter se teneant datam rationem ut $p:q$, sive ut sit

$$\frac{\partial P}{\partial Q} = \frac{p}{q}.$$

1) Vide notam p. 197. H. D.
2) Cf. Commentationem 779 huius voluminis. H. D.

4. Ista quidem conditio certo respectu facillime adimpleri potest, ita ut adeo relatio quaecunque inter quantitates P et Q statui possit. Namque si curva aq ita referat binas quantitates datas p et q, ut sumta abscissa $ap = p$, applicata pq fiat $= q$, super eodem axe construatur pro lubitu curva quaecunque AQ, ac sumto in priori curva puncto quocunque q, ductaque chorda aq, in altera curva capiatur punctum Q, ad quod ducta tangens QT illi cordae aq fiat parallela; quo facto coordinatae huius alterius curvae AP et PQ exhibebunt ipsas quantitates quaesitas P et Q. Si enim ponamus $AP = P$ et $PQ = Q$, in triangulo PQT utique erit $PQ : PT = \partial Q : \partial P$. Cum igitur hoc triangulum simile sit triangulo pqa, erit $\partial Q : \partial P = q : p$, quae est ipsa proportio requisita.



5. Verum haec constructio, licet facilis ac plana, ad institutum nostrum parum confert, propterea quod inventio puncti Q postulat resolutionem aequationum cuiusque ordinis, quae tamen neutiquam est in nostra potestate. Nam si natura curvae AQ hac tantum aequatione exprimatur:

$$Q = \alpha P + \beta P^2 + \gamma P^6,$$

hinc fiet

$$\frac{\partial Q}{\partial P} = \alpha + 2\beta P + 6\gamma P^5$$

fractioni $\frac{q}{p}$ aequalis statuenda, ita ut quantitas P erui debeat ex hac aequatione ordinis quinti:

$$\alpha + 2\beta P + 6\gamma P^5 = \frac{q}{p},$$

cuius resolutio utique vires Algebrae superat. Multo maiorem autem difficultatem offendemus, si aequatio inter P et Q magis fuerit complicata, in eaque

etiam altiores potestates ipsius Q occurrant; quamobrem ad inveniendas quantitates P et Q longe alia via nobis est ineunda.

6. Cum igitur haec aequatio resolvenda proponatur $\frac{\partial Q}{\partial P} = \frac{q}{p}$, ubi quantitates p et q ut datae spectantur, ponamus brevitatis gratia $\frac{q}{p} = t$, ut esse debeat

$$\frac{\partial Q}{\partial P} = t, \quad \text{ideoque} \quad \partial Q = t\partial P,$$

quae formula cum integrabilis esse debeat, ob

$$Q = \int t\partial P$$

per reductiones notissimas habebimus

$$Q = tP - \int P\,\partial t;$$

ita ut tantum formula $\int P\,\partial t$ integrabilis sit reddenda, id quod facillime praestatur, ponendo $\int P\,\partial t = T$. Hinc enim fiet

$$P = \frac{\partial T}{\partial t},$$

unde, quaecunque functio algebraica ipsius t pro T accipiatur, semper idoneum valorem pro quantitate P adipiscimur, scilicet

$$P = \frac{\partial T}{\partial t};$$

ex quo porro elicimus

$$Q = \frac{t\partial T}{\partial t} - T,$$

sicque plene satisfactum erit conditioni requisitae

$$\frac{\partial Q}{\partial P} = \frac{q}{p} = t.$$

Sumto enim elemento ∂t constanti, erit

$$\partial P = \frac{\partial\partial T}{\partial t} \quad \text{et} \quad \partial Q = \partial T + \frac{t\partial\partial T}{\partial t} - \partial T = \frac{t\partial\partial T}{\partial t},$$

unde manifesto prodit $\frac{\partial Q}{\partial P} = t$, uti requiritur. Eadem autem aequalitas prodiisset, etiamsi elementum ∂t non fuisset constans assumtum; tum enim prodiisset

$$\partial P = \frac{\partial t \partial \partial T - \partial T \partial \partial t}{\partial t^2}$$

et

$$\partial Q = \frac{\partial t^2 \partial T + t \partial t \partial \partial T - t \partial T \partial \partial t}{\partial t^2} - \partial T = \frac{t \partial t \partial \partial T - t \partial T \partial \partial t}{\partial t^2},$$

unde iterum colligitur $\frac{\partial Q}{\partial P} = t$, ut ante.

7. Inventis autem duabus quantitatibus P et Q ipsa formula differentialis quaesita ∂W duplici modo expressa habetur, scilicet

$$\text{vel } \partial W = \frac{\partial P}{p} \text{ vel } \partial W = \frac{\partial Q}{q},$$

quae autem necessario ad eandem expressionem deducere debent; ex utraque enim colligitur fore

$$\partial W = \frac{\partial t \partial \partial T - \partial T \partial \partial t}{p \partial t^2}.$$

Cum autem hic sit $t = \frac{q}{p}$, erit

$$\partial t = \frac{p \partial q - q \partial p}{pp};$$

unde cum sit

$$P = \frac{\partial T}{\partial t}, \quad \text{erit} \quad P = \frac{pp \partial T}{p \partial q - q \partial p},$$

unde reperimus

$$\partial P = \frac{pp \partial \partial T}{p \partial q - q \partial p} + \frac{2p \partial p \partial T}{p \partial q - q \partial p} - \frac{pp \partial T (p \partial \partial q - q \partial \partial p)}{(p \partial q - q \partial p)^2},$$

hincque denique ipsa formula differentialis quaesita erit

$$\partial W = \frac{p \partial \partial T}{p \partial q - q \partial p} + \frac{2 \partial p \partial T}{p \partial q - q \partial p} - \frac{p \partial T (p \partial \partial q - q \partial \partial p)}{(p \partial q - q \partial p)^2};$$

tum autem necessario fiet, uti constituimus

$$\int p \partial W = P = \frac{pp \partial T}{p \partial q - q \partial p}$$

et

$$\int q \partial W = Q = \frac{pq \partial T}{p \partial q - q \partial p} - T.$$

ALIA SOLUTIO

8. Quoniam ambo multiplicatores praescripti q et p aequaliter in computum ingredi debebant, quod tamen in solutione inventa longe secus evenit, ubi altera harum quantitatum p longe alia ratione inest atque altera q, operae pretium erit eiusmodi solutionem tradere, in quam ambae quantitates p et q pari ratione ingrediantur, ita ut, facta earum permutatione, formula pro ∂W inventa nullam alterationem patiatur, quandoquidem haec circumstantia ad elegantiam solutionis pertinere est censenda, licet solutio ante inventa in se spectata quaestioni pariter perfecte satisfaciat.

9. Maneant igitur in praecedente solutione omnia eadem usque ad introductionem litterae T; et quoniam pervenimus ad hanc aequationem:

$$Q = tP - \int P\partial t,$$

ubi ob $t = \frac{q}{p}$ est

$$\partial t = \frac{p\partial q - q\partial p}{pp},$$

quae formula denominatorem habet pp, statuamus

$$\int P\partial t = \frac{v}{p},$$

ut differentiando prodeat

$$\frac{P(p\partial q - q\partial p)}{pp} = \frac{p\partial v - v\partial p}{pp},$$

sicque obtinebimus

$$P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p},$$

ex quo valore porro deducimus

$$Q = \frac{q}{p}\cdot\frac{p\partial v - v\partial p}{p\partial q - q\partial p} - \frac{v}{p},$$

quae expressio reducitur ad hanc:

$$Q = \frac{q\partial v - v\partial q}{p\partial q - q\partial p},$$

quae alteri P perfecte est analoga, dum valor ipsius Q ex P sponte prodit permutatione litterarum p et q, solo signo excepto.

10. Cum igitur invenerimus

$$P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p},$$

per differentiationem nanciscimur

$$\partial P = \frac{(p\partial q - q\partial p)(p\partial\partial v - v\partial\partial p) - (p\partial v - v\partial p)(p\partial\partial q - q\partial\partial p)}{(p\partial q - q\partial p)^2},$$

quae expressio, facta evolutione, reducitur ad hanc:

$$\partial P = \frac{p\partial\partial v(p\partial q - q\partial p) - p\partial v(p\partial\partial q - q\partial\partial p) + pv(\partial p\partial\partial q - \partial q\partial\partial p)}{(p\partial q - q\partial p)^2}.$$

Hinc igitur formula differentialis quaesita ∂W ita exprimetur, ut sit

$$\partial W = \frac{\partial\partial v(p\partial q - q\partial p) - \partial v(p\partial\partial q - q\partial\partial p) + v(\partial p\partial\partial q - \partial q\partial\partial p)}{(p\partial q - q\partial p)^2};$$

ubi ambae quantitates p et q manifesto sunt permutabiles, si quidem mutatio signorum nullum discrimen afferre est censenda.

11. Ista solutio non solum antecedentem supereminet insigni elegantia, sed etiam pariter est maxime generalis, quandoquidem quantitas v arbitrio nostro penitus relinquitur; ideoque eius loco omnes plane functiones ipsarum p et q accipi possunt. At vero ista expressio conditiones praescriptas ita adimplet, ut inde fiat

$$\int p\partial W = P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p}$$

et

$$\int q\partial W = Q = \frac{q\partial v - v\partial q}{p\partial q - q\partial p}.$$

Quae ambae expressiones utique sunt algebraicae, dummodo pro v functiones algebraicae ipsarum p et q accipiantur.

12. Isti valores integrales nobis insuper duas insignes proprietates formulae differentialis inventae declarant, quae in eo consistunt, ut ista formula ad nihilum redigatur, tam posito $v = p$ quam $v = q$. Cum enim formula integralis

$\int p\,\partial W$ manifesto evanescat posito $v = p$, necesse est ut etiam formula differentialis eodem casu evanescat; quod idem de altera formula integrali est tenendum, quae casu $v = q$ evanescit.

COROLLARIUM

13. Quoniam solutio huius problematis eo est perducta, ut binae quantitates P et Q investigentur, quarum differentialia ∂P et ∂Q datam inter se teneant rationem ut $p:q$, operae pretium erit posteriorem solutionem sub forma theorematis memoriae imprimi.

THEOREMA

14. Si duae quantitates P et Q desiderentur, quarum differentialia ∂P et ∂Q eandem inter se teneant rationem quam duae quantitates p et q, ita ut esse debeat

$$\frac{\partial P}{\partial Q} = \frac{p}{q},$$

huic requisito generalissime satisfiet, sumendo

$$P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p} \quad \text{et} \quad Q = \frac{q\partial v - v\partial q}{p\partial q - q\partial p},$$

ubi quantitas v penitus arbitrio nostro est relicta.

EXEMPLUM 1

15. *Invenire formulam differentialem ∂W, quae tam per sinum quam per cosinum cuiuspiam anguli variabilis φ multiplicata evadat integrabilis.*

Hic ergo erit

$$p = \sin.\varphi \quad \text{et} \quad q = \cos.\varphi,$$

hincque differentiando (ubi quidem elementum $\partial\varphi$ constans assumamus) prodibit

$$\partial p = \partial\varphi \cos.\varphi \quad \text{et} \quad \partial q = -\partial\varphi \sin.\varphi$$

porroque

$$\partial\partial p = -\partial\varphi^2 \sin.\varphi \quad \text{et} \quad \partial\partial q = -\partial\varphi^2 \cos.\varphi,$$

unde formulae, quae in expressionem ipsius ∂W ingrediuntur, sequentes valores sortientur:

$$\begin{aligned} \text{I.}\quad & p\,\partial q - q\,\partial p = -\,\partial\varphi, \\ \text{II.}\quad & p\,\partial\partial q - q\,\partial\partial p = 0, \\ \text{III.}\quad & \partial p\,\partial\partial q - \partial q\,\partial\partial p = -\,\partial\varphi^3, \end{aligned}$$

ex quibus valoribus ergo concluditur formula differentialis quaesita

$$\partial W = -\frac{\partial\partial v}{\partial\varphi} - v\,\partial\varphi.$$

16. Quod haec formula prodiit negativa, negotium nullo modo turbat, ac tuto statuere poterimus

$$\partial W = v\,\partial\varphi + \frac{\partial\partial v}{\partial\varphi}$$

sumto scilicet elemento $\partial\varphi$ constante; tum autem pariter, mutatis signis, erit

$$\int \partial W \sin.\varphi = \frac{\partial v \sin.\varphi}{\partial\varphi} - v\cos.\varphi$$

et

$$\int \partial W \cos.\varphi = \frac{\partial v \cos.\varphi}{\partial\varphi} + v\sin.\varphi.$$

Tum vero etiam evidens est ipsam formulam ∂W evanescere, tam casu $v = \sin.\varphi$ quam casu $v = \cos.\varphi$.

17. Quodsi ergo formula ∂W exprimat elementum cuiuspiam lineae curvae ∂s, ut sit $\partial s = v\,\partial\varphi + \frac{\partial\partial v}{\partial\varphi}$, quaecunque functio algebraica fuerit v, semper curva algebraica exhiberi poterit. Constitutis enim coordinatis orthogonalibus x et y, si sumatur

$$\partial x = \partial s\cos.\varphi \quad \text{et} \quad \partial y = \partial s\sin.\varphi,$$

ut fiat

$$\partial x^2 + \partial y^2 = \partial s^2,$$

quia ambae hae formulae sunt integrabiles, coordinatae curvae quaesitae erunt

$$x = \frac{\partial v\cos.\varphi}{\partial\varphi} + v\sin.\varphi \quad \text{et} \quad y = \frac{\partial v\sin.\varphi}{\partial\varphi} - v\cos.\varphi,$$

ubi notasse iuvabit fore

$$xx + yy = \frac{\partial v^2}{\partial \varphi^2} + vv.$$

Praeterea vero, si formula $\int v \partial \varphi$ integrationem admittat, tum curva ipsa erit rectificabilis; fiet enim

$$s = \int v \partial \varphi + \frac{\partial v}{\partial \varphi}.$$

SCHOLION

18. Quo insignis usus huius tractationis uberius ob oculos ponatur, tam huic exemplo, quam sequentibus, cuique problema speciale adiungamus, in quo integratio cuiuspiam aequationis differentialis secundi gradus perficiatur; quod saepissime egregium usum habere poterit.

PROBLEMA SPECIALE 1

19. *Si Φ denotet functionem quamcunque ipsius φ, resolvere istam aequationem differentialem secundi gradus*

$$v \partial \varphi + \frac{\partial \partial v}{\partial \varphi} = \Phi \partial \varphi,$$

in qua elementum $\partial \varphi$ constans est assumtum, sive per integrationem invenire valorem ipsius v.

Quia iam invenimus huius aequationis membrum sinistrum integrabile fieri duobus casibus, dum vel per sin. φ vel cos. φ multiplicatur, membrum autem dextrum iam est functio ipsius φ tantum; eius integratio nulla laborat difficultate. Primo enim haec aequatio in sin. φ ducta et integrata dabit

$$\frac{\partial v \sin. \varphi}{\partial \varphi} - v \cos. \varphi = \int \Phi \partial \varphi \sin. \varphi;$$

at vero multiplicatio per cos. φ praebebit

$$\frac{\partial v \cos. \varphi}{\partial \varphi} + v \sin. \varphi = \int \Phi \partial \varphi \cos. \varphi.$$

20. Cum igitur iam geminam habeamus aequationem primi gradus, sine ulla ulteriori integratione valorem quantitatis v elicere poterimus: prior enim

in cos. φ ducta et a posteriori in sin. φ ducta ablata perducet ad hanc aequationem:

$$v = \sin.\varphi \int \Phi \partial\varphi \cos.\varphi - \cos.\varphi \int \Phi \partial\varphi \sin.\varphi,$$

quod integrale utique est completum, propterea quod, ob binas formulas integrales, geminam constantem arbitrariam involvit.

21. Quoniam per reductiones notissimas est

$$\int \Phi \partial\varphi \sin.\varphi = -\Phi \cos.\varphi + \int \partial\Phi \cos.\varphi$$

et

$$\int \Phi \partial\varphi \cos.\varphi = \Phi \sin.\varphi - \int \partial\Phi \sin.\varphi,$$

si hi valores substituantur, valor ipsius v etiam hoc modo exprimi poterit:

$$v = \Phi - \sin.\varphi \int \partial\Phi \sin.\varphi - \cos.\varphi \int \partial\Phi \cos.\varphi.$$

EXEMPLUM 2

22. *Invenire formulam differentialem ∂W, quae tam per tangentem, quam secantem cuiuspiam anguli variabilis φ multiplicata evadat integrabilis.*

Hic igitur esto $p = \text{tang.}\varphi$ et $q = \sec.\varphi$, unde sequitur

$$\partial p = \frac{\partial\varphi}{\cos.\varphi^2} \quad \text{et} \quad \partial q = \frac{\partial\varphi \sin.\varphi}{\cos.\varphi^2},$$

porro vero

$$\partial\partial p = \frac{2\partial\varphi^2 \sin.\varphi}{\cos.\varphi^3} \quad \text{et} \quad \partial\partial q = \frac{\partial\varphi^2}{\cos.\varphi} + \frac{2\partial\varphi^2 \sin.\varphi^2}{\cos.\varphi^3},$$

sive

$$\partial\partial q = \frac{2\partial\varphi^2}{\cos.\varphi^3} - \frac{\partial\varphi^2}{\cos.\varphi}.$$

Hinc igitur colliguntur sequentes aequationes:

$$\text{I.} \quad p\partial q - q\partial p = -\frac{\partial\varphi}{\cos.\varphi}.$$

$$\text{II.} \quad p\partial\partial q - q\partial\partial p = -\frac{\partial\varphi^2}{\cos.\varphi}\,\text{tang.}\varphi = -\frac{\partial\varphi^2 \sin.\varphi}{\cos.\varphi^2}.$$

$$\text{III.} \quad \partial p\partial\partial q - \partial q\partial\partial p = \frac{\partial\varphi^3}{\cos.\varphi^3}.$$

Ex quibus ergo valoribus concluditur formula differentialis quaesita

$$\partial W = -\frac{\partial\partial v \cos.\varphi}{\partial\varphi} + \partial v \sin.\varphi + \frac{v\partial\varphi}{\cos.\varphi}.$$

23. Ternos autem illos valores facilius hoc modo reperire licet. Primo enim cum sit

$$\frac{q}{p} = \frac{1}{\sin.\varphi},$$

erit differentiando

$$\frac{p\partial q - q\partial p}{pp} = -\frac{\partial\varphi \cos.\varphi}{\sin.\varphi^2},$$

unde per $pp = \frac{\sin.\varphi^2}{\cos.\varphi^2}$ multiplicando oritur

$$p\partial q - q\partial p = -\frac{\partial\varphi}{\cos.\varphi},$$

quae denuo differentiata dat

$$p\partial\partial q - q\partial\partial p = -\frac{\partial\varphi^2 \sin.\varphi}{\cos.\varphi^2}.$$

Deinde cum sit $\frac{\partial q}{\partial p} = -\sin.\varphi$, erit differentiando

$$\frac{\partial p\partial\partial q - \partial q\partial\partial p}{\partial p^2} = \partial\varphi \cos.\varphi,$$

quae per $\partial p^2 = \frac{\partial\varphi^2}{\cos.\varphi^4}$ multiplicata dat

$$\partial p\partial\partial q - \partial q\partial\partial p = \frac{\partial\varphi^3}{\cos.\varphi^3}.$$

Cum igitur sit

$$\partial W = -\frac{\partial\partial v \cos.\varphi}{\partial\varphi} + \partial v \sin.\varphi + \frac{v\partial\varphi}{\cos.\varphi},$$

erit

$$\int p\partial W = P = \frac{v}{\cos.\varphi} - \frac{\partial v \sin.\varphi}{\partial\varphi}$$

et

$$\int q\partial W = Q = \frac{v \sin.\varphi}{\cos.\varphi} - \frac{\partial v}{\partial\varphi}.$$

PROBLEMA SPECIALE 2

24. *Denotante Φ functionem quamcunque anguli φ, resolvere istam aequationem secundi gradus:*

$$-\frac{\partial\partial v \cos.\varphi}{\partial\varphi} + \partial v \sin.\varphi + \frac{v\partial\varphi}{\cos.\varphi} = \Phi\partial\varphi.$$

Hic primo per tang. φ multiplicando et integrando obtinetur haec aequatio primi gradus:

$$\frac{v}{\cos.\varphi} - \frac{\partial v \sin.\varphi}{\partial\varphi} = \int\Phi\partial\varphi \operatorname{tang.}\varphi;$$

at vero per sec. φ multiplicando et integrando prodit

$$\frac{v \sin.\varphi}{\cos.\varphi} - \frac{\partial v}{\partial\varphi} = \int\Phi\partial\varphi \sec.\varphi.$$

Haec posterior ducta in sin. φ et a priori subtracta dat

$$v\cos.\varphi = \int\Phi\partial\varphi \operatorname{tang.}\varphi - \sin.\varphi\int\Phi\partial\varphi \sec.\varphi$$

ideoque

$$v = \sec.\varphi\int\Phi\partial\varphi \operatorname{tang.}\varphi - \operatorname{tang.}\varphi\int\Phi\partial\varphi \sec.\varphi,$$

quae est integratio completa aequationis propositae.

25. Denique hic annotasse iuvabit, si φ denotet angulum, quem curvae cuiuspiam elementum ∂s cum elemento abscissae ∂x constituit, atque ∂W exprimat ipsum elementum abscissae ∂x, tum fore elementum applicatae

$$\partial y = \partial x \operatorname{tang.}\varphi = p\partial W,$$

elementum vero curvae

$$\partial s = \partial x \sec.\varphi = q\partial W,$$

unde ergo habebitur ipsa applicata

$$y = \frac{v}{\cos.\varphi} - \frac{\partial v \sin.\varphi}{\partial\varphi},$$

atque ipsa curvae longitudo erit

$$s = \frac{v \sin. \varphi}{\cos. \varphi} - \frac{\partial v}{\partial \varphi}.$$

Cum igitur sit

$$\partial x = -\frac{\partial \partial v \cos. \varphi}{\partial \varphi} + \partial v \sin. \varphi + \frac{v \partial \varphi}{\cos. \varphi},$$

si modo haec formula etiam integrationem admittat, tum prodibit curva algebraica, simulque rectificabilis. Iam vero integrando, qua fieri licet, prodit

$$x = -\frac{\partial v}{\partial \varphi} \cos. \varphi - \int \partial v \sin. \varphi + \int \partial v \sin. \varphi + \int \frac{v \partial \varphi}{\cos. \varphi} = -\frac{\partial v \cos. \varphi}{\partial \varphi} + \int \frac{v \partial \varphi}{\cos. \varphi}.$$

Quam ob rem necesse est, ut formula $\int \frac{v \partial \varphi}{\cos. \varphi}$ integrationem admittat, veluti evenit, si sumatur $v = \cos. \varphi^2$, tum enim erit

$$\int \frac{v \partial \varphi}{\cos. \varphi} = \sin. \varphi,$$

atque ob

$$\partial v = -2 \partial \varphi \sin. \varphi \cos. \varphi$$

prodibit

$$x = 2 \sin. \varphi \cos. \varphi^2 + \sin. \varphi = \sin. \varphi (1 + 2 \cos. \varphi^2),$$

tum vero erit

$$y = \cos. \varphi (1 + 2 \sin. \varphi^2)$$

et arcus curvae

$$s = 3 \sin. \varphi \cos. \varphi.$$

EXEMPLUM 3

26. *Invenire formulam differentialem ∂W, quae sive multiplicata sive divisa per datam quantitatem t evadat integrabilis.*

Hic ergo hae duae formulae:

$$t \partial W \quad \text{et} \quad \frac{\partial W}{t}$$

reddi debent integrabiles. Pro hoc igitur exemplo erit

$$p = t \quad \text{et} \quad q = \frac{1}{t},$$

unde fit

$$\partial p = \partial t \quad \text{et} \quad \partial q = -\frac{\partial t}{tt}.$$

Cum ergo sit $\frac{q}{p} = \frac{1}{tt}$, erit

$$\frac{p\partial q - q\partial p}{pp} = -\frac{2\partial t}{t^3},$$

unde per $pp = tt$ multiplicando fit

$$p\partial q - q\partial p = -\frac{2\partial t}{t};$$

quae formula porro differentiata, sumendo ∂t constans, praebet

$$p\partial\partial q - q\partial\partial p = +\frac{2\partial t^2}{tt}.$$

Deinde quia est $\frac{\partial q}{\partial p} = -\frac{1}{tt}$, fiet

$$\frac{\partial p\partial\partial q - \partial q\partial\partial p}{\partial p^2} = \frac{2\partial t}{t^3},$$

quae aequatio per ∂p^2 multiplicata praebet

$$\partial p\partial\partial q - \partial q\partial\partial p = \frac{2\partial t^3}{t^3},$$

ex quibus valoribus colligitur formula quaesita

$$\partial W = -\frac{t\partial\partial v}{2\partial t} - \tfrac{1}{2}\partial v + \frac{v\partial t}{2t}$$

sive hunc valorem duplicando et signa mutando statui potest

$$\partial W = \frac{t\partial\partial v}{\partial t} + \partial v - \frac{v\partial t}{t}.$$

27. Multiplicemus igitur hanc formam per t, ut fiat

$$t\partial W = \frac{tt\partial\partial v}{\partial t} + t\partial v - v\partial t,$$

ubi primi membri integratio dat

$$\int t\partial W = \frac{tt\partial v}{\partial t} - \int (t\partial v + v\partial t),$$

sicque prodit

$$\int t\partial W = \frac{tt\partial v}{\partial t} - tv.$$

Simili modo cum sit

$$\frac{\partial W}{t} = \frac{\partial\partial v}{\partial t} + \frac{\partial v}{t} - \frac{v\partial t}{tt},$$

erit integrando

$$\int\frac{\partial W}{t} = \frac{\partial v}{\partial t} + \int\frac{(t\partial v - v\partial t)}{tt} = \frac{\partial v}{\partial t} + \frac{v}{t}.$$

PROBLEMA SPECIALE 3

28. *Denotante T functionem quamcunque ipsius t, resolvere aequationem secundi gradus*

$$\frac{t\partial\partial v}{\partial t} + \partial v - \frac{v\partial t}{t} = T\partial t.$$

Haec resolutio per praecedentia facile expeditur; primo enim per t multiplicando et integrando prodit

$$\frac{tt\partial v}{\partial t} - tv = \int Tt\partial t.$$

At vero per t dividendo et integrando fit

$$\frac{\partial v}{\partial t} + \frac{v}{t} = \int\frac{T\partial t}{t},$$

unde eliminando terminum $\frac{\partial v}{\partial t}$ oritur

$$v = \tfrac{1}{2}t\int\frac{T\partial t}{t} - \frac{1}{2t}\int Tt\partial t,$$

quae expressio quomodo satisfaciat, videamus. Primo hinc erit

$$\frac{\partial v}{\partial t} = \tfrac{1}{2}\int\frac{T\partial t}{t} + \frac{1}{2tt}\int Tt\partial t,$$

quae aequatio denuo differentiata dat

$$\frac{\partial\partial v}{\partial t^2} = \frac{T}{t} - \frac{1}{t^3}\int Tt\partial t.$$

His igitur valoribus substitutis prodit

$$
\begin{array}{rl}
\dfrac{t\partial\partial v}{\partial t} = & T\partial t - \dfrac{\partial t}{tt}\displaystyle\int T t\partial t \\
+\,\partial v = & \dfrac{\partial t}{2}\displaystyle\int \dfrac{T\partial t}{t} + \dfrac{\partial t}{2tt}\int T t\partial t \\
-\dfrac{v\partial t}{t} = & -\dfrac{\partial t}{2}\displaystyle\int \dfrac{T\partial t}{t} + \dfrac{\partial t}{2tt}\int T t\partial t \\
\hline
\text{Summa} = & T\partial t, \text{ prorsus ut ante.}
\end{array}
$$

SCHOLION

29. Duo priora exempla, quae hic attulimus, insignem usum praestant in curvarum indole respectu rectificationis exploranda, uti ostendimus; tertium autem exemplum ideo est notatu dignum, quod in huiusmodi investigationibus saepius eiusmodi formulae differentiales occurrunt, quas, per eandem quantitatem tam multiplicatas quam divisas, reddi oportet integrabiles. Veluti si in superficie cylindri recti praeter rectas axi parallelas aliae lineae duci debeant, quae sint rectificabiles, quaestio ad inventionem eiusmodi quantitatis algebraicae t reducitur, per quam ista formula differentialis $\frac{\partial v}{\sqrt{(1-vv)}}$ tam multiplicata quam divisa integrationem admittat. Postquam autem plurimum nequicquam in hoc negotio desudassem, asseverare non dubito, nullam plane dari eiusmodi quantitatem t, qua hae duae formulae:

$$\frac{t\partial v}{\sqrt{(1-vv)}} \quad \text{et} \quad \frac{\partial v}{t\sqrt{(1-vv)}}$$

simul fiant integrabiles. Praeterea vero omnino certum mihi videtur, praeter simplices potestates ipsius v nullas alias eius functiones t dari, unde hae duae formulae differentiales: $\frac{t\partial v}{v}$ et $\frac{\partial v}{tv}$ simul evadant integrabiles.

PROBLEMA

30. *Si formula differentialis ∂W utcunque fuerit composita ex quantitatibus variabilibus p et q (inter quas quidem certa relatio dari assumitur), definire quantitatem v ex hac aequatione differentiali secundi gradus:*

$$\partial W = \frac{\partial\partial v(p\partial q - q\partial p) - \partial v(p\partial\partial q - q\partial\partial p) + v(\partial p\partial\partial q - \partial q\partial\partial p)}{(p\partial q - q\partial p)^2}.$$

SOLUTIO

Quia novimus hanc aequationem integrabilem reddi, si multiplicetur tam per p quam per q, haec duplex integratio nobis duas suppeditat aequationes differentiales primi gradus, quae sunt:

$$\int p\,\partial W = \frac{p\partial v - v\partial p}{p\partial q - q\partial p} \quad \text{et} \quad \int q\,\partial W = \frac{q\partial v - v\partial q}{p\partial q - q\partial p},$$

quarum posterior, ducta in p, si subtrahatur a priore ducta in q, praebet sequentem aequationem:

$$q\int p\,\partial W - p\int q\,\partial W = v;$$

sicque innotescit valor quaesitus quantitatis v, qui, quoniam geminam integrationem involvit, ob duplicem constantem arbitrariam pro integrali completo aequationis differentio-differentialis est habendus.

PROBLEMA 2

Invenire formulam differentialem ∂W, quae per tres quantitates variabiles datas p, q et r multiplicata fiat integrabilis.

SOLUTIO

31. Ponamus haec tria integralia, quae prodire debent, esse:

$$1^0.\ \int p\,\partial W = P, \quad 2^0.\ \int q\,\partial W = Q, \quad 3^0.\ \int r\,\partial W = R,$$

unde triplici modo formula differentialis quaesita ∂W exprimetur

$$1^0.\ \partial W = \frac{\partial P}{p}, \quad 2^0.\ \partial W = \frac{\partial Q}{q}, \quad \text{et} \quad 3^0.\ \partial W = \frac{\partial R}{r}.$$

32. Iam supra § 14 vidimus duabus prioribus conditionibus, quibus requiritur, ut sit $\frac{\partial P}{\partial Q} = \frac{p}{q}$, satisfieri, si capiatur

$$P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p} \quad \text{et} \quad Q = \frac{q\partial v - v\partial q}{p\partial q - q\partial p}.$$

Simili modo conditionibus primae et tertiae, qua requiritur, ut sit $\frac{\partial P}{\partial R} = \frac{p}{r}$, satisfiet, loco v aliam quantitatem u scribendo, his duobus valoribus:

$$P = \frac{p\partial u - u\partial p}{p\partial r - r\partial p} \quad \text{et} \quad R = \frac{r\partial u - u\partial r}{p\partial r - r\partial p}.$$

Nunc igitur totum negotium eo est reductum, ut ambo valores pro P inventi ad aequalitatem perducantur; ubi ergo quaeritur, quales quantitates pro v et u accipi debeant, ut istae duae formulae pro P inventae inter se evadant aequales, quippe quo facto simul etiam bini reliqui valores Q et R innotescent.

33. Cum igitur debeat esse

$$\frac{p\partial v - v\partial p}{p\partial q - q\partial p} = \frac{p\partial u - u\partial p}{p\partial r - r\partial p},$$

quo hoc facilius effici possit, statuamus

$$v = Vp \quad \text{et} \quad u = Up,$$

et conditio adimplenda erit

$$\frac{\partial V}{p\partial q - q\partial p} = \frac{\partial U}{p\partial r - r\partial p}, \quad \text{ideoque} \quad \frac{\partial V}{\partial U} = \frac{p\partial q - q\partial p}{p\partial r - r\partial p}.$$

Quare si theorema supra § 14 datum in subsidium vocemus, loco P et Q nunc habemus V et U, at vero loco p et q nunc habemus $p\,\partial q - q\,\partial p$ et $p\,\partial r - r\,\partial p$, ideoque loco v introducendo quantitatem Z haec conditio adimplebitur, si statuamus:

$$V = \frac{(p\partial q - q\partial p)\partial Z - Z(p\partial\partial q - q\partial\partial p)}{(p\partial q - q\partial p)(p\partial\partial r - r\partial\partial p) - (p\partial r - r\partial p)(p\partial\partial q - q\partial\partial p)},$$

$$U = \frac{(p\partial r - r\partial p)\partial Z - Z(p\partial\partial r - r\partial\partial p)}{(p\partial q - q\partial p)(p\partial\partial r - r\partial\partial p) - (p\partial r - r\partial p)(p\partial\partial q - q\partial\partial p)},$$

ubi notetur denominatorem revocari posse ad sequentem formam satis concinnam:

$$p\,\partial\partial p(q\,\partial r - r\,\partial q) + p\,\partial\partial q(r\,\partial p - p\,\partial r) + p\,\partial\partial r(p\,\partial q - q\,\partial p),$$

in qua praeter factorem communem p ternae litterae p, q et r sunt inter se permutabiles.

34. Solutio ergo nostri problematis sequenti modo absolvetur:

1°. Sumta pro lubitu quantitate quacunque variabili Z, quaeratur quantitas V, ut sit

$$Vp = \frac{(p\partial q - q\partial p)\partial Z - Z(p\partial\partial q - q\partial\partial p)}{\partial\partial p(q\partial r - r\partial q) + \partial\partial q(r\partial p - p\partial r) + \partial\partial r(p\partial q - q\partial p)},$$

qui simul est valor litterae v.

2°. Eodem modo colligatur valor

$$Up = \frac{(p\partial r - r\partial p)\partial Z - Z(p\partial\partial r - r\partial\partial p)}{\partial\partial p(q\partial r - r\partial q) + \partial\partial q(r\partial p - p\partial r) + \partial\partial r(p\partial q - q\partial p)},$$

qui simul est valor ipsius u.

3°. Inventis autem his valoribus pro v et u formentur porro isti valores:

$$P = \frac{p\partial v - v\partial p}{p\partial q - q\partial p}, \text{ vel etiam } P = \frac{p\partial u - u\partial p}{p\partial r - r\partial p},$$

quippe qui ambo valores ad aequalitatem sunt perducti; praeterea vero sumatur

$$Q = \frac{q\partial v - v\partial q}{p\partial q - q\partial p} \text{ et } R = \frac{r\partial u - u\partial r}{p\partial r - r\partial p}.$$

4°. Denique hae ternae formulae $\frac{\partial P}{p}$, $\frac{\partial Q}{q}$, $\frac{\partial R}{r}$ producent expressiones inter se prorsus aequales, atque adeo valorem formulae differentialis quaesitae ∂W.

35. Solutio haec adhuc magis contrahi potest, si, positis ut ante $v = Vp$ et $u = Up$, insuper statuatur $q = px$ et $r = py$, ubi quia q et r tanquam functiones ipsius p spectari possunt, etiam x et y erunt functiones cognitae ipsius p; tum autem fiet $\frac{\partial V}{\partial U} = \frac{\partial x}{\partial y}$. Quare cum x et y sint functiones ipsius p, ponatur $\partial x = X\partial p$ et $\partial y = Y\partial p$, ita ut etiam X et Y futurae sint functiones cognitae ipsius p, hinc autem habebimus hanc aequationem resolvendam $\frac{\partial V}{\partial U} = \frac{X}{Y}$.

36. Nunc igitur introducendo novam quantitatem variabilem indefinitam Z theorema nostrum § 14 nobis dabit

$$V = \frac{X\partial Z - Z\partial X}{X\partial Y - Y\partial X} \text{ et } U = \frac{Y\partial Z - Z\partial Y}{X\partial Y - Y\partial X}.$$

Inventis autem valoribus V et U simul habentur litterae $v = Vp$ et $u = Up$, ex quibus ut ante determinabuntur valores P, Q et R, hincque tandem ipsa formula differentialis quaesita ∂W.

COROLLARIUM

37. Quoniam litterae V et U duabus constant partibus, altera per Z, altera vero per ∂Z affecta, etiam quantitates v et u duabus huiusmodi partibus constabunt; unde earum differentialia insuper partem secundo differentiali $\partial\partial Z$ affectam continebunt. Huiusmodi ergo tres partes in litteris P, Q et R occurrent, quae cum denuo differentiari debeant, ut formula differentialis ∂W eliciatur, evidens est in expressione ∂W differentialia ipsius Z ad tertium gradum assurgere, unde pro ∂W huiusmodi prodibit expressio:

$$\partial W = AZ + B\partial Z + C\partial\partial Z + D\partial^3 Z,$$

ubi litteras A, B, C et D pro formulis per evolutionem valorum supra assignatorum posuimus. Quibus praeceptis applicationi ad casus particulares non immorari est opus.

METHODUS SINGULARIS RESOLVENDI AEQUATIONES DIFFERENTIALES SECUNDI GRADUS

M. S. Academiae exhibuit die 21. Ianuarii 1779

Commentatio 677 indicis ENESTROEMIANI

Institutiones calculi integralis 4, 1794, p. 525—533

1. Si p et q fuerint functiones quaecunque ipsius x, atque proposita fuerit haec aequatio inter binas variabiles x et z:

$$2p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{z\partial x}{q},$$

evidens est eius integrale facile inveniri posse, si ea multiplicetur per z, ut habeatur

$$2pz\partial z + zz\partial p = \frac{z\partial x}{q}\int\frac{z\partial x}{q}.$$

Prioris enim membri integrale est pzz, posterius vero membrum posito

$$\int\frac{z\partial x}{q} = v$$

abit in $v\partial v$, cuius integrale est $\frac{1}{2}(vv + C)$, ita ut hinc nanciscamur istam aequationem integralem:

$$pzz = \tfrac{1}{2}(vv + C),$$

unde fit

$$vv = 2pzz - C,$$

hincque

$$v = \int \frac{z\partial x}{q} = \sqrt{(2pzz - C)},$$

quae differentiata dat

$$\frac{z\partial x}{q} = \frac{2pz\partial z + zz\partial p}{\sqrt{(2pzz - C)}},$$

facta ergo divisione per z, erit

$$\frac{\partial x}{q} = \frac{2p\partial z + z\partial p}{\sqrt{(2pzz - C)}},$$

quemadmodum autem hinc valor ipsius z per x, eiusque functiones p et q exprimi queat, non liquet. Ut autem istum scopum obtineamus, posito ut fecimus

$$\int \frac{z\partial x}{q} = v\,, \quad \text{ut sit} \quad vv = 2pzz - C\,,$$

retineamus quantitatem v in calculo, et cum sit

$$\frac{z\partial x}{q} = \partial v, \quad \text{erit} \quad z = \frac{q\partial v}{\partial x},$$

quo valore substituto habebimus

$$vv = \frac{2pqq\partial v^2}{\partial x^2} - C,$$

unde colligitur

$$\partial v = \frac{\partial x \sqrt{(vv + C)}}{q\sqrt{2p}},$$

quae sponte separationem admittit, cum sit

$$\frac{\partial v}{\sqrt{(vv + C)}} = \frac{\partial x}{q\sqrt{2p}},$$

ideoque

$$\int \frac{\partial v}{\sqrt{(vv + C)}} = \int \frac{\partial x}{q\sqrt{2p}},$$

cuius valor, quoniam p et q sunt functiones ipsius x, tanquam cognitus spectari potest.

2. Statuamus ergo hoc integrale

$$\int \frac{\partial x}{q\sqrt{2p}} = lX,$$

ut habeamus

$$\int \frac{\partial v}{\sqrt{(vv+C)}} = lX,$$

quare cum constet esse

$$\int \frac{\partial v}{\sqrt{(vv+C)}} = l[v+\sqrt{(vv+C)}],$$

erit

$$v+\sqrt{(vv+C)} = X,$$

unde colligitur

$$v = \frac{X^2 - C}{2X},$$

ideoque per quantitatem X definitur.

3. Cum igitur supra invenerimus $2pzz = vv + C$, erit

$$2pzz = \frac{(X^2-C)^2}{4XX} + C = \frac{(XX+C)^2}{4XX},$$

consequenter erit

$$z\sqrt{2p} = \frac{X^2+C}{2X},$$

sicque quantitas z ita per X exprimitur, ut sit

$$z = \frac{X^2+C}{2X\sqrt{2p}},$$

ubi meminisse oportet esse

$$lX = \int \frac{\partial x}{q\sqrt{2p}}, \quad \text{sive} \quad X = e^{\int \frac{\partial x}{q\sqrt{2p}}}.$$

4. Manifestum autem est, aequationem nostram propositam, si a signo integrali liberetur, abire in aequationem differentialem secundi gradus, cuius ergo integrale completum modo elicuimus. Facta enim multiplicatione per q fiet

$$2pq\partial z + qz\partial p = \partial x \int \frac{z\partial x}{q},$$

et differentiatio sumto elemento ∂x constante praebebit sequentem aequationem differentialem secundi gradus:

$$2pq\partial\partial z + 2p\partial q\partial z + z\partial q\partial p + 3q\partial p\partial z + qz\partial\partial p = \frac{z\partial x^2}{q},$$

cuius ergo aequationis non parum abstrusae novimus esse integrale completum

$$z = \frac{X^2 + C}{2X\sqrt{2p}}, \quad \text{existente} \quad X = e^{\int\frac{\partial x}{q\sqrt{2p}}},$$

ita ut ista quantitas X etiam constantem arbitrariam involvat.

5. Cum autem haec aequatio non parum sit complicata, sequenti modo concinnius repraesentari potest; cum enim sit

$$\frac{q}{z}\partial\cdot pzz = \partial x\int\frac{z\partial x}{q},$$

erit differentiationem tantum indicando

$$\partial\cdot\frac{q\partial\cdot pzz}{z} = \frac{z\partial x^2}{q},$$

quae manifesto integrabilis evadit, si multiplicetur per $\frac{2q\partial\cdot pzz}{z}$, quodsi enim brevitatis gratia statuatur

$$\frac{q\partial\cdot pzz}{z} = s\partial x,$$

membrum sinistrum fit

$$2s\partial x\partial\cdot s\partial x = 2s\partial s\partial x^2,$$

eiusque ergo integrale $ss\partial x^2$; at vero ex parte dextra habebimus $2\partial x^2\partial\cdot pzz$, cuius igitur integrale est

$$2pzz\partial x^2 + C\partial x^2,$$

ita ut integratio nobis praebeat

$$ss = 2pzz + C.$$

6. Quo nunc hanc aequationem penitius evolvamus, statuamus ut ante $pzz = v$[1]), ita ut sit

$$\frac{q\partial v}{z} = s\partial x,$$

1) v hoc loco scriptum non eundem numerum denotat ac qui scribitur § 1, 2, 3. H. D.

eritque nostrum integrale inventum

$$ss = \frac{qq\partial v^2}{zz\partial x^2} = 2v + C,$$

quae ob $zz = \frac{v}{p}$ abit in hanc:

$$\frac{pqq\partial v^2}{v\partial x^2} = 2v + C,$$

unde eruitur propemodum ut ante

$$\frac{\partial v}{\sqrt{v(2v+C)}} = \frac{\partial x}{q\sqrt{p}},$$

quae a forma ante inventa non discrepat.

7. Simili modo etiam aliae aequationes differentiales magis complicatae resolvi poterunt, veluti si proponatur ista aequatio:

$$3p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{zz\partial x}{q},$$

ubi iterum p et q denotant functiones quascunque ipsius x. Cum enim sit

$$3p\partial z + z\partial p = \frac{\partial\cdot pz^3}{zz},$$

erit per zz multiplicando

$$\partial\cdot pz^3 = \frac{zz\partial x}{q}\int\frac{zz\partial x}{q},$$

quae posito $\int\frac{zz\partial x}{q} = v$ abit in $\partial\cdot pz^3 = v\partial v$, ideoque integrando

$$2pz^3 = vv + C.$$

8. Quoniam autem posuimus

$$\int\frac{zz\partial x}{q} = v,$$

erit $zz = \frac{q\partial v}{\partial x}$, hincque

$$z^3 = \frac{q\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}},$$

unde fit

$$\frac{2pq\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}} = vv + C.$$

Sumtis ergo quadratis erit

$$\frac{4ppq^3\partial v^3}{\partial x^3} = (vv + C)^2,$$

ideoque

$$\frac{\partial v^3}{(vv + C)^2} = \frac{\partial x^3}{4ppq^3},$$

cuius radix cubica praebet

$$\frac{\partial v}{\sqrt[3]{(vv + C)^2}} = \frac{\partial x}{q\sqrt[3]{4pp}}.$$

Hinc igitur quantitas v per x definitur, ita ut iam v spectare queamus tanquam veram functionem ipsius x, qua inventa erit

$$z^3 = \frac{vv + C}{2p}, \quad \text{hincque} \quad z = \sqrt[3]{\frac{vv + C}{2p}}.$$

9. Eadem ista aequatio adhuc alio modo resolvi poterit, quandoquidem per q multiplicata ita repraesentatur:

$$\frac{q\partial \cdot pz^3}{zz} = \partial x \int \frac{zz\partial x}{q},$$

sive

$$\partial \cdot \frac{q\partial \cdot pz^3}{zz} = \frac{zz\partial x^2}{q},$$

quae manifesto integrabilis redditur, multiplicando per $\frac{2q\partial \cdot pz^3}{zz}$, prodit enim

$$\left(\frac{q\partial \cdot pz^3}{zz}\right)^2 = 2pz^3\partial x^2 + C\partial x^2.$$

10. Iam ponatur $pz^3 = v$, ita ut sit

$$z^3 = \frac{v}{p}, \quad \text{et} \quad z^4 = \frac{v}{p}\sqrt[3]{\frac{v}{p}},$$

quo valore substituto habebimus

$$\frac{pqq\partial v^2\sqrt[3]{p}}{v\sqrt[3]{v}} = 2v\partial x^2 + C\partial x^2,$$

unde concluditur

$$\frac{\partial v^2}{v(2v+C)\sqrt[3]{v}} = \frac{\partial x^2}{pqq\sqrt[3]{p}},$$

sive

$$\frac{\partial v}{\sqrt{v(2v+C)\sqrt[3]{v}}} = \frac{\partial v}{v^{\frac{2}{3}}\sqrt{(2v+C)}} = \frac{\partial x}{q\sqrt[3]{pp}};$$

haec aequatio simplicior evadit, ponendo $v = u^3$, scilicet

$$\frac{3\partial u}{\sqrt{(2u^3+C)}} = \frac{\partial x}{q\sqrt[3]{pp}}.$$

Hinc intelligitur, innumerabilia exempla per has formulas expediri posse.

11. Quin etiam huiusmodi aequationes multo generaliores tractari poterunt; namque aequatio generalior ita potest repraesentari:

$$\frac{\partial \cdot pz^m}{z^n} = \frac{\partial x}{q}\int\frac{z^n\partial x}{q},$$

quae evoluta dat

$$mpz^{m-n-1}\partial z + z^{m-n}\partial p = \frac{\partial x}{q}\int\frac{z^n\partial x}{q}.$$

Facta autem multiplicatione per z^n, prodit aequatio sponte integrabilis

$$\partial\cdot pz^m = \frac{z^n\partial x}{q}\int\frac{z^n\partial x}{q},$$

si quidem prodit

$$2pz^m = \left(\int\frac{z^n\partial x}{q}\right)^2 + C.$$

12. Ad hanc aequationem ulterius evolvendam statuamus

$$\int\frac{z^n\partial x}{q} = v, \quad \text{eritque} \quad z^n = \frac{q\partial v}{\partial x},$$

unde primo

$$2pz^m = vv + C,$$

et hinc porro

$$(2p)^{\frac{n}{m}}\cdot z^n = (2p)^{\frac{n}{m}}\cdot\frac{q\partial v}{\partial x} = (vv+C)^{\frac{n}{m}},$$

quae cum sponte sit separabilis, dabit

$$\frac{\partial v}{(vv+C)^{\frac{n}{m}}} = \frac{\partial x}{q(2p)^{\frac{n}{m}}},$$

unde ergo quantitas v per x determinabitur, qua inventa ipsa quantitas quaesita z ita exprimetur, ut sit

$$z^m = \frac{vv+C}{2p}.$$

13. Illustremus haec unico exemplo a primo casu petito, sumendo scilicet $p = 1 + xx$ et $q = \sqrt{2}$, ita ut aequatio proposita sit

$$2\partial z(1+xx) + 2zx\partial x = \frac{\partial x}{2}\int z\partial x,$$

quae in hanc aequationem secundi gradus evolvitur:

$$4\partial\partial z(1+xx) + 12x\partial x\partial z + 3z\partial x^2 = 0,$$

cuius ergo integrale quaeritur.

14. Faciamus ergo applicationem solutionis supra § 3 inventae, ubi cum hic sit

$$p = 1 + xx \quad \text{et} \quad q = \sqrt{2},$$

erit

$$lX = \tfrac{1}{2}\int\frac{\partial x}{\sqrt{(1+xx)}} = \tfrac{1}{2}l[x+\sqrt{(1+xx)}] - \tfrac{1}{2}la,$$

unde fit

$$X = \frac{\sqrt{[x+\sqrt{(1+xx)}]}}{\sqrt{a}},$$

hoc igitur valore substituto habebimus

$$z = \frac{aC + x + \sqrt{(1+xx)}}{2\sqrt{2a(1+xx)[x+\sqrt{(1+xx)}]}},$$

quae hoc modo simplicius exprimitur

$$z = \frac{[aC + x + \sqrt{(1+xx)}]\sqrt{[-x+\sqrt{(1+xx)}]}}{2\sqrt{2a}(1+xx)}.$$

Ubi ergo duae quantitates constantes arbitrariae sunt involutae, atque adeo hoc integrale completum algebraice determinetur. Posito ergo $C = 0$, integrale particulare erit ex prima forma petitum

$$z = \frac{\sqrt{[x + \sqrt{(1 + xx)}]}}{2\sqrt{2a(1 + xx)}}.$$

15. Aliud integrale particulare hinc exhiberi potest, constantes ita sumendo, ut sit aC infinitum, at vero $C\sqrt{a}$ finitum $= b$, tum enim erit

$$z = \frac{aC}{2\sqrt{2a(1 + xx)[x + \sqrt{(1 + xx)}]}} = \frac{b}{2\sqrt{2(1 + xx)[x + \sqrt{(1 + xx)}]}},$$

quae forma redigitur ad hanc

$$z = \frac{\alpha\sqrt{[-x + \sqrt{(1 + xx)}]}}{\sqrt{(1 + xx)}}.$$

METHODUS NOVA INVESTIGANDI OMNES CASUS QUIBUS HANC AEQUATIONEM DIFFERENTIO-DIFFERENTIALEM $\partial\partial y(1-axx)-bx\partial x\partial y-cy\partial x^2=0$ RESOLVERE LICET

M. S. Academiae exhibuit die 13. Ianuarii 1780

Commentatio 678 indicis ENESTROEMIANI

Institutiones calculi integralis 4, 1794, p. 533—543

1.[1]) Hic quidem in usum vocari posset methodus a me[2]) et ab aliis iam passim exposita, qua valor ipsius y per seriem infinitam exprimitur. Tunc enim omnibus casibus, quibus haec series alicubi abrumpitur, habebitur integrale particulare aequationis propositae; unde quidem haud difficulter integrale completum erui poterit. Verum etsi hoc modo infiniti casus integrabiles reperiuntur, tamen non omnes innotescunt, sed dantur praeterea infiniti alii casus, qui resolutionem admittunt. Quamobrem hic methodum prorsus singularem proponam, cuius ope omnes plane casus integrabiles elici poterunt. Haec autem methodus ita est comparata, ut cognito casu quocunque resolutionem admittente, ex eo innumerabiles alii deduci queant.

2. Statim autem se offerunt duo casus simplicissimi, quibus resolutio succedit, quorum alter est, si $c=0$, alter vero, si $b=a$, quos ergo binos casus principales ante omnia evolvi oportet.

1) In editione principe paragraphi numeris 16 usque ad 35 signatae sunt. H. D.

2) Vide *Institutionum calculi integralis* vol. 2⁵, § 967—992. *LEONHARDI EULERI Opera omnia*, series I, vol. 12. H. D.

CASUS PRIOR PRINCIPALIS QUO $c = 0$

3. Hoc igitur casu aequatio nostra erit

$$\partial\partial y(1 - axx) = bx\partial x\partial y,$$

quae posito $\partial y = p\partial x$, abit in hanc:

$$\partial p(1 - axx) = bpx\partial x,$$

sive

$$\frac{\partial p}{p} = \frac{bx\partial x}{1 - axx},$$

cuius integrale est

$$lp = -\frac{b}{2a}l(1 - axx) + lC,$$

sicque erit

$$p = C(1 - axx)^{-\frac{b}{2a}} = \frac{\partial y}{\partial x},$$

unde obtinetur

$$y = C\int\partial x(1 - axx)^{-\frac{b}{2a}};$$

ubi notasse iuvabit istum valorem fieri algebraicum, quoties fuerit $-\frac{b}{2a}$ numerus integer positivus, sive $b = -2ia$ denotante i numerum integrum quemcunque. Tum vero valor integralis etiam algebraicus evadit, quando fuerit $-\frac{b}{2a}$, sive $-\frac{3}{2}$, sive $-\frac{5}{2}$, sive $-\frac{7}{2}$ etc. ideoque in genere $\frac{b}{a} = 2i + 1$, ubi esse nequit $i = 0$.

CASUS PRINCIPALIS ALTER QUO $b = a$

4. Hoc ergo casu aequatio nostra per $2\partial y$ multiplicata erit

$$2\partial y\partial\partial y(1 - axx) - 2ax\partial x\partial y^2 - 2cy\partial y\partial x^2 = 0,$$

quae sponte est integrabilis, eius enim integrale erit

$$\partial y^2(1 - axx) - cyy\partial x^2 = C\partial x^2.$$

Ex hac igitur aequatione erit

$$\partial y \sqrt{(1 - axx)} = \partial x \sqrt{(C + cyy)},$$

separatione ergo facta erit

$$\frac{\partial x}{\sqrt{(1 - axx)}} = \frac{\partial y}{\sqrt{(C + cyy)}}.$$

In hac ergo forma iterum continentur casus algebraici, ad quos eruendos faciamus $a = -\alpha\alpha$, $c = \gamma\gamma$ et $C = \beta\beta$, ut habeamus

$$\frac{\partial x}{\sqrt{(1 + \alpha\alpha xx)}} = \frac{\partial y}{\sqrt{(\beta\beta + \gamma\gamma yy)}},$$

cuius integrale est

$$\frac{1}{\alpha} l[\alpha x + \sqrt{(1 + \alpha\alpha xx)}] = \frac{1}{\gamma} l[\gamma y + \sqrt{(\beta\beta + \gamma\gamma yy)}] - \frac{1}{\gamma} l\Delta,$$

unde ad numeros ascendendo erit

$$\gamma y + \sqrt{(\beta\beta + \gamma\gamma yy)} = \Delta[\alpha x + \sqrt{(1 + \alpha\alpha xx)}]^{\frac{\gamma}{\alpha}}.$$

Posito ergo V pro hac expressione posteriore erit

$$V - \gamma y = \sqrt{(\beta\beta + \gamma\gamma yy)},$$

et sumtis quadratis

$$y = \frac{VV - \beta\beta}{2\gamma V}.$$

Cum igitur sit

$$V = \Delta[\alpha x + \sqrt{(1 + \alpha\alpha xx)}]^{\frac{\gamma}{\alpha}},$$

erit

$$2\gamma y = \Delta[\alpha x + \sqrt{(1 + \alpha\alpha xx)}]^{\frac{\gamma}{\alpha}} - \frac{\beta\beta}{\Delta}[\alpha x + \sqrt{(1 + \alpha\alpha xx)}]^{-\frac{\gamma}{\alpha}},$$

ubi est $\beta\beta = C$, exponens vero $\frac{\gamma}{\alpha} = \sqrt{-\frac{c}{a}}$, sicque, quoties $\sqrt{-\frac{c}{a}}$ fuerit numerus rationalis, integrale semper erit algebraicum.

5. His duobus casibus principalibus expeditis duplicem tradam viam aequationem propositam in infinitas alias eiusdem generis transformandi, ita ut semper aequatio huius formae

$$\partial\partial Y(1-axx)-Bx\partial x\partial Y-CY\partial x^2=0$$

prodeat, quae cum resolutionem admittat casibus vel $C=0$ vel $B=a$, iisdem casibus etiam ipsa aequatio proposita erit resolubilis. Duplices igitur hasce transformationes iam sum expositurus.

TRANSFORMATIONES PRIORIS ORDINIS

6. Statuo $y=\frac{\partial v}{\partial x}$, unde ob

$$\partial y=\frac{\partial\partial v}{\partial x} \quad \text{et} \quad \partial\partial y=\frac{\partial^3 v}{\partial x}$$

aequatio nostra induet hanc formam:

$$\partial^3 v(1-axx)-bx\partial x\partial\partial v-c\partial x^2\partial v=0,$$

cuius singuli termini integrationem admittunt; erit enim

$$\int\partial x^2\partial v=v\partial x^2,$$
$$\int x\partial x\partial\partial v=x\partial x\partial v-v\partial x^2,$$
$$\int\partial^3 v(1-axx)=\partial\partial v(1-axx)+2ax\partial x\partial v-2av\partial x^2.$$

His partibus colligendis, aequatio nostra erit

$$\partial\partial v(1-axx)-(b-2a)x\partial x\partial v-(c-b+2a)v\partial x^2=0,$$

quae cum propositae prorsus sit similis, integrabilis erit his duobus casibus $c-b+2a=0$ et $b=3a$, sive quoties fuerit $c=b-2a$ vel $b=3a$, atque integratione pro utroque casu instituta, ita ut v exprimatur per x, tum pro ipsa aequatione proposita erit $y=\frac{\partial v}{\partial x}$; unde patet, si integralia pro v inventa fuerint algebraica, fore quoque valorem ipsius y algebraicum.

7. Quod si ulterius simili modo statuamus $v=\frac{\partial v'}{\partial x}$, quoniam per operationem praecedentem litterae b et c transibunt in $b-2a$ et $c-b+2a$, nunc ista aequatio proveniet

$$\partial\partial v'(1-axx)-(b-4a)x\partial x\partial v'-(c-2b+6a)v'\partial x^2=0,$$

quae ergo integrabilis erit, si fuerit vel $b=5a$ vel $c=2b-6a$. Atque inventis valoribus pro v' fiet $y=\frac{\partial\partial v'}{\partial x^2}$, scilicet differentialia secunda ipsius v' dabunt y sicque, si pro v' valor algebraicus prodierit, etiam y adipiscetur valorem algebraicum.

8. Quod si eandem substitutionem denuo repetamus ponendo $v'=\frac{\partial v''}{\partial x}$, pro litteris initialibus b et c iam habebimus $b-6a$ et $c-3b+12a$, et aequatio resultans erit

$$\partial\partial v''(1-axx)-(b-6a)x\partial x\partial v''-(c-3b+12a)v''\partial x^2=0,$$

quae ergo resolutionem admittet, quoties fuerit vel $b=7a$ vel $c=3b-12a$, quibus ergo casibus etiam ipsa aequatio proposita resolutionem admittat necesse est, cum sit $y=\frac{\partial^3 v''}{\partial x^3}$.

9. Quod si ergo easdem has operationes continuo repetamus, perpetuo ad aequationes eiusdem formae perveniemus; ubi notasse sufficiet ambos valores, quos pro litteris b et c in qualibet operatione obtinuerimus, quos una cum valoribus ipsius y in sequenti tabula ob oculos ponamus

	b	c	y
Operatio I.	$b-2a$	$c-b+2a$	$\frac{\partial v}{\partial x}$
II.	$b-4a$	$c-2b+6a$	$\frac{\partial\partial v'}{\partial x^2}$
III.	$b-6a$	$c-3b+12a$	$\frac{\partial^3 v''}{\partial x^3}$
IV.	$b-8a$	$c-4b+20a$	$\frac{\partial^4 v'''}{\partial x^4}$
—	- - -	- - - - - - -	- -
—	- - -	- - - - - - -	- -
—	- - -	- - - - - - -	- -
i	$b-2ia$	$c-ib+i(i+1)a$	$\frac{\partial^i v^{(i-1)}}{\partial x^i}$

10. Hinc igitur in genere patet, aequationem propositam semper resolutionem admittere, quoties fuerit

$$\text{vel } b = 2ia + a, \quad \text{vel } c = ib - i(i + 1)a,$$

ubi pro i omnes numeros integros positivos accipere licet, ita ut hinc duos ordines innumerabilium casuum integrabilium nanciscamur, quorum posteriores tantum per methodum serierum initio indicatam reperiuntur, priores vero huic methodo prorsus sint inaccessi.

TRANSFORMATIONES POSTERIORIS ORDINIS

11. Quemadmodum hic per differentialia sumus progressi, nunc per integralia regrediamur, ac primo quidem ponamus $y = \int z\,\partial x$, et aequatio proposita evadet

$$\partial z(1 - axx) - bxz\,\partial x - c\,\partial x \int z\,\partial x = 0,$$

quae differentiata ad formam propositam reducitur

$$\partial\partial z(1 - axx) - (b + 2a)x\,\partial x\,\partial z - (c + b)z\,\partial x^2 = 0,$$

quae ergo secundum casus principales integrationem admittet, casibus $c + b = 0$ et $b + 2a = a$, sive $c = -b$ et $b = -a$. Integralibus igitur inventis erit $y = \int z\,\partial x$; unde patet, etiamsi haec integralia fuerint algebraica, tamen valores ipsius y fieri transcendentes.

12. Simili modo statuamus porro $z = \int z'\,\partial x$, et quia per praecedentem operationem loco b et c adepti sumus $b + 2a$ et $c + b$, nunc perveniemus ad hanc aequationem

$$\partial\partial z'(1 - axx) - (b + 4a)x\,\partial x\,\partial z' - (c + 2b + 2a)z'\,\partial x^2 = 0,$$

quae ergo integrationem admittet, si fuerit vel $c + 2b + 2a = 0$, vel $b + 4a = a$, sive $c = -2b - 2a$ et $b = -3a$. Integralibus autem hinc inventis pro y habebimus $y = \int \partial x \int z'\,\partial x$, quae ita ad signum integrale simplex reducitur, ut sit

$$y = x\int z'\,\partial x - \int z' x\,\partial x.$$

13. Simili modo statuamus porro $z'=\int z''\partial x$, atque nunc deducemur ad hanc aequationem:

$$\partial\partial z''(1-axx)-(b+6a)x\partial x\partial z''-(c+3b+6a)z''\partial x^2=0,$$

quae igitur integrabilis erit, si fuerit vel $c+3b+6a=0$, vel $b+6a=a$, hoc est si $c=-3b-6a$ et $b=-5a$; atque ex his integralibus fiet $y=\int\partial x\int\partial x\int z''\partial x$, qui valor ex praecedente reduci potest, si is per ∂x multiplicatus denuo integretur et loco z' scribatur z''; obtinetur enim

$$y=\tfrac{1}{2}xx\int z''\partial x-x\int xz''\partial x+\tfrac{1}{2}\int xxz''\partial x.$$

14. Quod si iam has operationes ulterius continuemus, totum negotium huc redibit, ut formulae, quae loco b et c sunt proditurae, rite formentur, simulque valores ipsius y assignentur, quemadmodum sequens tabula indicabit:

	b	c	y
Operat. I.	$b+2a$	$c+b$	$\int z\partial x$
II.	$b+4a$	$c+2b+2a$	$\int\partial x\int z'\partial x$
III.	$b+6a$	$c+3b+6a$	$\int\partial x\int\partial x\int z''\partial x$
IV.	$b+8a$	$c+4b+12a$	$\int\partial x\int\partial x\int\partial x\int z'''\partial x$
—	—	—	—
—	—	—	—
—	—	—	—
i	$b+2ia$	$c+ib+i(i-1)a$	$\int\partial x\int\partial x\cdots\int z^{(i-1)}\partial x$

15. Ex antecedentibus satis manifestum est, quomodo integralia ista complicata ad simplicia reduci queant, unde tantum sequentem tabulam subiungemus:

$$\int\partial x\int z'\partial x=x\int z'\partial x-\int z'x\partial x$$
$$\int\partial x\int\partial x\int z''\partial x=\tfrac{1}{2}(xx\int z''\partial x-2x\int z''x\partial x+\int z''xx\partial x)$$
$$\int\partial x\int\partial x\int\partial x\int z'''\partial x=\tfrac{1}{6}(x^3\int z'''\partial x-3xx\int z'''x\partial x+3x\int z'''xx\partial x-\int z'''x^3\partial x)$$
$$\int\partial x\int\partial x\int\partial x\int\partial x\int z^{IV}\partial x=\tfrac{1}{24}(x^4\int z^{IV}\partial x-4x^3\int z^{IV}x\partial x+6xx\int z^{IV}xx\partial x-4x\int z^{IV}x^3\partial x+\int z^{IV}x^4\partial x)$$

etc.

16. Quod si iam has operationes secundum numerum indefinitum i continuemus, et loco b, c, z scribamus B, C, Z, aequatio resultans erit

$$\partial\partial Z(1-axx)-Bx\partial x\partial Z-CZ\partial x^2=0,$$

ubi erit, uti iam indicavimus,

$$B=b+2ia \quad \text{et} \quad C=c+ib+i(i-1)a;$$

quamobrem haec aequatio integrationem admittet, quoties fuerit

$$\text{vel } C=0 \quad \text{hoc est} \quad c=-ib-i(i-1)a,$$
$$\text{vel } B=a \quad \text{hoc est} \quad b=-(2i-1)a;$$

quae formulae ab illis, quas supra pro priori transformationum ordine invenimus, tantum in hoc discrepant, quod hic littera i valorem negativum accepit; unde adiungatur sequens

CONCLUSIO GENERALIS

17. Si littera i hic denotet omnes numeros integros sive positivos sive negativos, aequatio proposita differentio-differentialis

$$\partial\partial y(1-axx)-bx\partial x\partial y-cy\partial x^2=0$$

semper integrationem sive resolutionem admittet, quoties fuerit

$$1^0.)\ c=ib-i(i+1)a,$$

vel

$$2^0.)\ b=(2i+1)a,$$

ubi asseverare licet[1]), omnes plane casus resolubiles in hac duplici forma contineri, ita ut nullus plane casus integrationem admittens exhiberi queat, qui non in alterutra harum duarum formularum comprehendatur, dum contra methodus per series procedens, cuius initio mentionem fecimus, tantum casus integrabiles priores ostendit, ita ut inde infinitus numerus casuum pariter resolubilium inde excludatur[2]).

1) Quod non est hic demonstratum. H. D.

2) Cf. L. Euleri Commentationem 274: *Constructio aequationis differentio-differentialis*

$$Aydu^2+(B+Cu)dudy+(D+Eu+Fuu)ddy=0,$$

sumto elemento du constante. Novi Comment. acad. sc. Petrop. 8, 1763, p. 150. *Leonhardi Euleri Opera omnia*, series I, vol. 22, p. 395. H. D.

13. Simili modo statuamus porro $z'=\int z''\partial x$, atque nunc deducemur ad hanc aequationem:

$$\partial\partial z''(1-axx)-(b+6a)x\partial x\partial z''-(c+3b+6a)z''\partial x^2=0,$$

quae igitur integrabilis erit, si fuerit vel $c+3b+6a=0$, vel $b+6a=a$, hoc est si $c=-3b-6a$ et $b=-5a$; atque ex his integralibus fiet $y=\int\partial x\int\partial x\int z''\partial x$, qui valor ex praecedente reduci potest, si is per ∂x multiplicatus denuo integretur et loco z' scribatur z''; obtinetur enim

$$y=\tfrac{1}{2}xx\int z''\partial x-x\int xz''\partial x+\tfrac{1}{2}\int xxz''\partial x.$$

14. Quod si iam has operationes ulterius continuemus, totum negotium huc redibit, ut formulae, quae loco b et c sunt prodituræ, rite formentur, simulque valores ipsius y assignentur, quemadmodum sequens tabula indicabit:

	b	c	y
Operat. I.	$b+2a$	$c+b$	$\int z\partial x$
II.	$b+4a$	$c+2b+2a$	$\int\partial x\int z'\partial x$
III.	$b+6a$	$c+3b+6a$	$\int\partial x\int\partial x\int z''\partial x$
IV.	$b+8a$	$c+4b+12a$	$\int\partial x\int\partial x\int\partial x\int z'''\partial x$
—	—	—	—
—	—	—	—
—	—	—	—
i	$b+2ia$	$c+ib+i(i-1)a$	$\int\partial x\int\partial x\cdots\int z^{(i-1)}\partial x$

15. Ex antecedentibus satis manifestum est, quomodo integralia ista complicata ad simplicia reduci queant, unde tantum sequentem tabulam subiungemus:

$\int\partial x\int z'\partial x=x\int z'\partial x-\int z'x\partial x$
$\int\partial x\int\partial x\int z''\partial x=\tfrac{1}{2}(xx\int z''\partial x-2x\int z''x\partial x+\int z''xx\partial x)$
$\int\partial x\int\partial x\int\partial x\int z'''\partial x=\tfrac{1}{6}(x^3\int z'''\partial x-3xx\int z'''x\partial x+3x\int z'''xx\partial x-\int z'''x^3\partial x)$
$\int\partial x\int\partial x\int\partial x\int\partial x\int z^{IV}\partial x=\tfrac{1}{24}(x^4\int z^{IV}\partial x-4x^3\int z^{IV}x\partial x+6xx\int z^{IV}xx\partial x-4x\int z^{IV}x^3\partial x+\int z^{IV}x^4\partial x$

etc.

16. Quod si iam has operationes secundum numerum indefinitum i continuemus, et loco b, c, z scribamus B, C, Z, aequatio resultans erit

$$\partial\partial Z(1 - axx) - Bx\partial x\partial Z - CZ\partial x^2 = 0,$$

ubi erit, uti iam indicavimus,

$$B = b + 2ia \quad \text{et} \quad C = c + ib + i(i-1)a;$$

quamobrem haec aequatio integrationem admittet, quoties fuerit

$$\begin{aligned} &\text{vel } C = 0 \quad \text{hoc est} \quad c = -ib - i(i-1)a, \\ &\text{vel } B = a \quad \text{hoc est} \quad b = -(2i-1)a; \end{aligned}$$

quae formulae ab illis, quas supra pro priori transformationum ordine invenimus, tantum in hoc discrepant, quod hic littera i valorem negativum accepit; unde adiungatur sequens

CONCLUSIO GENERALIS

17. Si littera i hic denotet omnes numeros integros sive positivos sive negativos, aequatio proposita differentio-differentialis

$$\partial\partial y(1 - axx) - bx\partial x\partial y - cy\partial x^2 = 0$$

semper integrationem sive resolutionem admittet, quoties fuerit

$$1^0.)\ c = ib - i(i+1)a,$$

vel

$$2^0.)\ b = (2i+1)a,$$

ubi asseverare licet[1]), omnes plane casus resolubiles in hac duplici forma contineri, ita ut nullus plane casus integrationem admittens exhiberi queat, qui non in alterutra harum duarum formularum comprehendatur, dum contra methodus per series procedens, cuius initio mentionem fecimus, tantum casus integrabiles priores ostendit, ita ut inde infinitus numerus casuum pariter resolubilium inde excludatur[2]).

1) Quod non est hic demonstratum. H. D.

2) Cf. L. Euleri Commentationem 274: *Constructio aequationis differentio-differentialis*

$$Aydu^2 + (B + Cu)dudy + (D + Eu + Fuu)ddy = 0,$$

sumto elemento du constante. Novi Comment. acad. sc. Petrop. 8, 1763, p. 150. *Leonhardi Euleri Opera omnia,* series I, vol. 22, p. 395. H. D.

COROLLARIUM 1

18. Transformetur aequatio proposita in aequationem differentialem primi gradus ponendo $y=e^{\int u\partial x}$, ac perveniemus ad hanc aequationem:

$$\partial u+uu\partial x-\frac{bux\partial x+c\partial x}{1-axx}=0,$$

quae ergo etiam integrationem admittet casibus quibus

$$\text{vel } b=(2i+1)a \text{ vel } c=ib-i(i+1)a,$$

denotante i numerum quemcunque integrum sive positivum sive negativum.

COROLLARIUM 2

19. Quod si porro ponatur

$$u=(1-axx)^n v,$$

posito brevitatis gratia $n=-\frac{b}{2a}$, pervenietur ad hanc aequationem ad genus RICCATIANUM referendam:

$$(1-axx)^n\partial v+(1-axx)^{2n}vv\partial x=\frac{c\partial x}{1-axx},$$

quae per $(1-axx)^n$ divisa abit in hanc:

$$\partial v+(1-axx)^n vv\partial x=\frac{c\partial x}{(1-axx)^{n+1}},$$

quae ergo iisdem casibus integrationem admittet.

COROLLARIUM 3

20. Quod si sumamus $a=0$, orietur ista aequatio:

$$\partial u+uu\partial x=bux\partial x+c\partial x,$$

quae ergo integrabilis erit, si fuerit vel $b = 0$ vel $c = ib$, quorum quidem prior casus per se est manifestus, quia tum erit

$$\partial x = \frac{\partial u}{c - uu}.$$

Haec forma autem commodius exprimi poterit, ponendo

$$u = \tfrac{1}{2}bx + v, \text{ unde } \partial v + vv\partial x = (c - \tfrac{1}{2}b)\partial x + \tfrac{1}{4}bbxx\partial x,$$

sive ponendo $b = 2f$, ut fiat

$$\partial v + vv\partial x = (c - f)\partial x + ffxx\partial x,$$

eritque haec aequatio integrabilis, quoties fuerit $c = 2if$, ita ut sequens aequatio semper integrationem admittat:

$$\partial v + vv\partial x = (2i - 1)f\partial x + ffxx\partial x,$$

quicunque numerus integer sive positivus sive negativus pro i accipiatur; hoc est, si in penultimo termino f multiplicetur per numerum imparem quemcunque sive positivum sive negativum, qui casus eo erunt abstrusiores, quo maior accipiatur numerus i; atque adeo vix alia via patere videtur integralia eruendi, nisi ut ad aequationem differentialem secundi gradus propositam regrediamur atque easdem operationes instituamus, quas supra docuimus. Interim tamen observavi, omnes istos casus etiam immediate ex ipsa aequatione per fractiones continuas derivari posse. Si enim proposita fuerit haec aequatio:

$$\partial v + vv\partial x = g\partial x + ffxx\partial x,$$

valor ipsius v duplici modo per fractionem continuam exprimi potest. Est enim priori modo

$$v = fx + \cfrac{g - f}{2fx + \cfrac{g - 3f}{2fx + \cfrac{g - 5f}{2fx + \text{etc.}}}}$$

Altero vero modo est

$$v = -fx - \cfrac{(g-f)}{2fx + g + 3f - \cfrac{}{2fx + g + 5f - \cfrac{}{2fx + g + 7f - \cfrac{}{2fx + \text{ etc.},}}}}$$

quarum prior abrumpitur, quoties fuerit $g = (2i + 1)f$, posterior vero, quoties fuerit $g = -(2i + 1)f$, qui sunt ipsi casus integrabiles ante inventi[1]).

1) Vide L. EULERI Commentationem 95: *De aequationibus differentialibus quae certis tantum casibus integrationem admittunt.* Comment. acad. sc. Petrop. 10, 1747, § 18 = *LEONHARDI EULERI Opera omnia*, series I, vol. 22, p. 162. Vide notam 2) p. 246. H.D.

DE FORMULIS INTEGRALIBUS IMPLICATIS EARUMQUE EVOLUTIONE ET TRANSFORMATIONE

M. S. Academiae exhibuit die 20. Aprilis 1778

Commentatio 679 indicis ENESTROEMIANI

Institutiones calculi integralis 4, 1794, p. 544—563

1.[1]) Talium formularum implicatarum forma generalis ita exhiberi potest:

$$\int p\,\partial x \int q\,\partial x \int r\,\partial x \int s\,\partial x \quad \text{etc.},$$

ubi quodvis signum integrale omnia sequentia in se complectitur. Ita ad valorem huius expressionis inveniendum a fine est incipiendum, positoque integrali $\int s\,\partial x = S$ erit

$$\int r\,\partial x \int s\,\partial x = \int S r\,\partial x,$$

cuius valor si ponatur $= R$, erit

$$\int q\,\partial x \int r\,\partial x \int s\,\partial x = \int R q\,\partial x,$$

quod integrale si ponatur $= Q$, valor ipsius formulae propositae erit $= \int Q p\,\partial x$, ubi per se intelligitur, in qualibet integratione more solito constantem arbitrariam in calculum introduci posse.

2. Hic scilicet probe tenendum est, istam expressionem $\int p\,\partial x \int q\,\partial x$ non significare productum ex formula $\int p\,\partial x$ in formulam $\int q\,\partial x$, sed integrale quod oritur, si tota formula differentialis $p\,\partial x \int q\,\partial x$ integretur; at vero si velimus

1) In editione principe paragraphi numeris 36 usque ad 79 signatae sunt. H. D.

productum talium duarum formularum integralium designare, id interpositione puncti fieri solet hoc modo $\int p\,\partial x \cdot \int q\,\partial x$, ubi scilicet punctum declarat praecedentia signa integralia non ultra hunc terminum extendi debere, ita haec forma:

$$\int p\,\partial x \int q\,\partial x \cdot \int r\,\partial x \int s\,\partial x$$

exprimit productum, quod oritur, si formula $\int p\,\partial x \int q\,\partial x$ multiplicetur per $\int r\,\partial x \int s\,\partial x$.

3. Hic igitur signandi modus non[1]) prorsus contrarius usu est receptus, atque in formulis differentialibus observari solet, ubi talis expressio $\partial x\,\partial y\,\partial z$ denotat productum trium differentialium ∂x, ∂y et ∂z, ita ut singula signa differentiationis tantum litteras immediate sequentes afficiant; at si velimus verbi gratia differentiale huius expressionis $x\,\partial y\,\partial z$ exprimere, hoc interpositione puncti fieri solet $\partial \cdot x\,\partial y\,\partial z$, ubi punctum significat, praefixum ∂ complecti totam expressionem sequentem.

4. Tales autem formulae integrales implicatae potissimum nascuntur ex continua integratione aequationum integralium linearium[2]), quarum forma in genere est

$$pz + \frac{q\,\partial z}{\partial x} + \frac{r\,\partial\partial z}{\partial x^2} + \frac{s\,\partial^3 z}{\partial x^3} + \text{etc.} = X,$$

ubi litterae p, q, r, s etc. sunt functiones datae variabilis x, cuius etiam functio quaecunque sit littera X, altera vero variabilis z ubique unam tantum tenet dimensionem; prouti haec forma generalis hic exhibetur, ad ordinem tertium differentialium refertur, ideoque ternas integrationes postulat, totidemque constantes arbitrarias involvere est censenda; hic scilicet ad methodum integrandi maxime naturalem respicio, quae per ternas integrationes successivas integrale desideratum producat.

5. Tali scilicet aequatione proposita ante omnia nosse oportet multiplicatorem, quo ea reddatur integrabilis, quem ergo supponamus esse $= \partial P$, atque integratione peracta prodeat ista aequatio:

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1) Editio princeps: signandi nos prorsus. H. D.

2) Cf. L. EULERI Commentationem 188: *Methodus aequationes differentiales altiorum graduum integrandi ulterius promota.* Novi Comment. acad. sc. Petrop. 3, 1753, p. 15. Cf. quoque *Institutiones calculi integralis,* vol. II, § 851—867, 865—872, 1138—1168, 1172—1194, 1126—1231, 1138—1272. LEONHARDI EULERI *Opera omnia,* series I, vol. 22, p. 181, vol. 12. Vide porro Commentationem 720 § 17, p. 346 huius voluminis. H. D.

$$p'z + \frac{q'\partial z}{\partial x} + \frac{r'\partial\partial z}{\partial x^2} = \int X\partial P,$$

quae aequatio iam est ordinis secundi; quodsi iam ponamus huius multiplicatorem idoneum esse $= \partial P'$, facta integratione oriatur haec aequatio primi ordinis, quae sit

$$p''z + \frac{q''\partial z}{\partial x} = \int \partial P' \int X\partial P,$$

pro qua si $\partial P''$ fuerit multiplicator idoneus, completum integrale induet hanc formam

$$p'''z = \int \partial P'' \int \partial P' \int X\partial P.$$

Sicque quantitas z exprimetur per formulam integralem implicatam.

6. Tali autem forma pro integrali inventa praecipuum negotium huc redit, ut ea ita evolvatur, ut formula continens functionem indefinitam X, quae hic terna signa integralia habet praefixa, plus unico ante se non habeat, quamobrem quemadmodum talis reductio commodissime institui queat, hic ostendere constitui, siquidem nisi certa artificia adhibeantur, huiusmodi operatio calculos maxime molestos postularet.

7. In genere autem huiusmodi formulas implicatas ita repraesentemus:

$$\int \partial p \int \partial q \int \partial r \int \partial s \int \partial t \quad \text{etc.},$$

pro cuius evolutione a casu duorum signorum integralium inchoemus, et quia erit $\int \partial p \int \partial q = \int q\,\partial p$, reductio vulgaris dat $pq - \int p\,\partial q$. Iam loco p et q iterum scribamus $\int \partial p$ et $\int \partial q$, atque evolutio ita se habebit:

$$\int \partial p \int \partial q = \int \partial p \cdot \int \partial q - \int \partial q \int \partial p,$$

ubi in genere hanc aequalitatem notasse iuvabit:

$$\int \partial p \int \partial q - \int \partial p \cdot \int \partial q + \int \partial q \int \partial p = 0.$$

8. Consideremus nunc formulam tria signa integralia involventem $\int \partial p \int \partial q \int \partial r$, et quia ut modo vidimus est

$$\int \partial q \int \partial r = qr - \int q\,\partial r,$$

nostra formula in has partes discerpitur $\int qr\,\partial p - \int\partial p\int q\,\partial r$, quae posterior pars reducitur ad hanc formam $p\int q\,\partial r - \int pq\,\partial r$, sicque formula nostra erit $\int qr\,\partial p - p\int q\,\partial r + \int pq\,\partial r$. Quoniam nunc requiritur, ut elementum ∂r in singulis partibus unicum tantum signum integrale habeat praefixum, ponamus $q\,\partial p = \partial v$, ut sit

$$v = \int q\,\partial p = \int\partial p\int\partial q,$$

eritque

$$\int qr\,\partial p = \int r\,\partial v = rv - \int v\,\partial r,$$

hincque colligitur

$$\int pq\,\partial r - \int v\,\partial r = \int\partial r(pq - v) = \int\partial r\int p\,\partial q.$$

Iam loco litterarum finitarum differentialia rursus introducentur, atque valor quaesitus formulae $\int\partial p\int\partial q\int\partial r$ sequenti modo exprimetur:

$$\int\partial p\int\partial q\cdot\int\partial r - \int\partial p\cdot\int\partial r\int\partial q + \int\partial r\int\partial q\int\partial p,$$

ubi in singulis membris elemento ∂r unicum signum integrale est praefixum.

9. Inter terna igitur elementa ∂p, ∂q et ∂r sequentem relationem notari operae erit pretium:

$$\int\partial p\int\partial q\int\partial r - \int\partial p\int\partial q\cdot\int\partial r + \int\partial p\cdot\int\partial r\int\partial q - \int\partial r\int\partial q\int\partial p = 0,$$

quodsi autem similem reductionem pro casibus plurium signorum integralium exsequi vellemus, in calculos molestissimos ac taediosissimos delaberemur; interim tamen totum hoc negotium per sequentia theoremata facillime et planissime expedietur, et quoniam singula membra ope puncti in duos factores resolvi convenit, ubi talis factor deest, eius locum unitate supplebimus.

THEOREMA 1

10. Pro unico elemento ∂p haec relatio habetur $\int\partial p\cdot 1 - 1\cdot\int\partial p = 0$, maxime obvia.

THEOREMA 2

11. Inter bina elementa ∂p et ∂q semper locum habebit haec relatio

$$\int\partial p\int\partial q\cdot 1 - \int\partial p\cdot\int\partial q + 1\cdot\int\partial q\int\partial p = 0.$$

DEMONSTRATIO

Ad hoc demonstrandum sufficiet ostendisse, differentiale huius aequationis esse $= 0$, quoniam vero singula membra binis constant factoribus, seorsim considerentur differentialia ex factoribus prioribus et posterioribus oriunda, hic igitur ex factoribus prioribus oritur differentiale

$$\partial p(\int\partial q \cdot 1 - 1 \cdot \int\partial q) = 0$$

per theorema 1. At ex factoribus posterioribus oritur differentiale

$$-\partial q(\int\partial p \cdot 1 - 1 \cdot \int\partial p) = 0.$$

THEOREMA 3

12. Inter terna elementa ∂p, ∂q et ∂r semper haec relatio locum habet:

$$\int\partial p \int\partial q \int\partial r \cdot 1 - \int\partial p \int\partial q \cdot \int\partial r + \int\partial p \cdot \int\partial r \int\partial q - 1 \cdot \int\partial r \int\partial q \int\partial p = 0.$$

DEMONSTRATIO

Hic iterum seorsim perpendantur differentialia tam ex prioribus quam ex posterioribus factoribus oriunda; ex prioribus autem oritur

$$\partial p(\int\partial q \int\partial r \cdot 1 - \int\partial q \cdot \int\partial r + 1 \cdot \int\partial r \int\partial q),$$

cuius valor manifesto ad nihilum redigitur per theorema 2, si scilicet litterae p et q uno gradu promoveantur; tum vero differentiale ex factoribus posterioribus ortum est

$$-\partial r(\int\partial p \int\partial q \cdot 1 - \int\partial p \cdot \int\partial q + 1 \cdot \int\partial q \int\partial p),$$

cuius valor pariter per theorema praecedens evanescit; quoniam igitur ambo differentialia sunt $= 0$, etiam ipsa forma nihilo vel etiam constanti aequalis esse debet, evidens autem est constantem sponte involvi in signis integralibus.

THEOREMA 4

13. Inter quaterna elementa ∂p, ∂q, ∂r et ∂s semper ista relatio locum habet:

$$\left.\begin{array}{c}\int\partial p \int\partial q \int\partial r \int\partial s \cdot 1 - \int\partial p \int\partial q \int\partial r \cdot \int\partial s + \int\partial p \int\partial q \cdot \int\partial s \int\partial r \\ - \int\partial p \cdot \int\partial s \int\partial r \int\partial q + 1 \cdot \int\partial s \int\partial r \int\partial q \int\partial p\end{array}\right\} = 0.$$

DEMONSTRATIO

Differentiatio factorum priorum suppeditat sequentem expressionem:

$$\partial p(\int\partial q\int\partial r\int\partial s\cdot 1-\int\partial q\int\partial r\cdot\int\partial s+\int\partial q\cdot\int\partial s\int\partial r-1\cdot\int\partial s\int\partial r\int\partial q),$$

quae ob theorema praecedens ad nihilum reducitur. Simili modo differentiatio factorum posteriorum praebet hanc expressionem

$$-\partial s(\int\partial p\int\partial q\int\partial r\cdot 1-\int\partial p\int\partial q\cdot\int\partial r+\int\partial p\cdot\int\partial r\int\partial q-1\cdot\int\partial r\int\partial q\int\partial p),$$

quae ob theorema 3 iterum est $=0$.

THEOREMA 5

14. Inter quina elementa ∂p, ∂q, ∂r, ∂s et ∂t semper haec relatio locum habet:

$$\left.\begin{array}{l}\int\partial p\int\partial q\int\partial r\int\partial s\int\partial t\cdot 1-\int\partial p\int\partial q\int\partial r\int\partial s\cdot\int\partial t+\int\partial p\int\partial q\int\partial r\cdot\int\partial t\int\partial s\\ -\int\partial p\int\partial q\cdot\int\partial t\int\partial s\int\partial r+\int\partial p\cdot\int\partial t\int\partial s\int\partial r\int\partial q-1\cdot\int\partial t\int\partial s\int\partial r\int\partial q\int\partial p\end{array}\right\}=0.$$

DEMONSTRATIO

Huius theorematis demonstratio prorsus eodem modo se habet ac theorematum praecedentium; sicque clarissime iam est evictum tales relationes perpetuo veritati esse consentaneas, quotcunque etiam elementis fuerint composita.

15. Quo vis horum theorematum clarius perspiciatur, operae pretium erit, ea per exempla determinata illustrasse; ponamus igitur esse

$$\partial p=x^{\alpha-1}\partial x,\ \ \partial q=x^{\beta-1}\partial x,\ \ \partial r=x^{\gamma-1}\partial x,\ \ \partial s=x^{\delta-1}\partial x,\ \ \partial t=x^{\varepsilon-1}\partial x,$$

atque ex theoremate primo statim aequatio identica nascitur $\frac{x^\alpha}{\alpha}-\frac{x^\alpha}{\alpha}=0.$ Verum theorema secundum nobis praebet hanc aequationem

$$\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)}-\frac{x^{\alpha+\beta}}{\alpha\beta}+\frac{x^{\alpha+\beta}}{\alpha(\alpha+\beta)}=0,$$

unde per $x^{\alpha+\beta}$ dividendo prodit haec aequalitas

$$\frac{1}{\beta(\alpha+\beta)}-\frac{1}{\alpha\beta}+\frac{1}{\alpha(\alpha+\beta)}=0,$$

cuius veritas satis facile in oculos incurrit.

16. Hae porro positiones in theoremate tertio introductae producent hanc aequationem

$$\frac{x^{\alpha+\beta+\gamma}}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)}-\frac{x^{\alpha+\beta+\gamma}}{\beta\gamma(\alpha+\beta)}+\frac{x^{\alpha+\beta+\gamma}}{\alpha\beta(\beta+\gamma)}-\frac{x^{\alpha+\beta+\gamma}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)},$$

unde per $x^{\alpha+\beta+\gamma}$ dividendo prodit haec egregia aequalitas:

$$\frac{1}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)}-\frac{1}{\beta\gamma(\alpha+\beta)}+\frac{1}{\alpha\beta(\beta+\gamma)}-\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)}=0.$$

17. Hae positiones iterum in theoremate quarto substitutae dant hanc aequationem:

$$\left.\begin{array}{l}\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}+\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}\\ \quad -\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}+\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)},\end{array}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta}$ divisa producit hanc aequationem:

$$\left.\begin{array}{l}\dfrac{1}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\dfrac{1}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)}+\dfrac{1}{\gamma\beta(\beta+\alpha)(\gamma+\delta)}\\ \quad -\dfrac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}+\dfrac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta}\end{array}\right\}=0.$$

18. Denique eaedem positiones in theoremate quinto substitutae producunt hanc aequationem:

$$\left.\begin{array}{l}\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha}) \\ \quad -\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}+\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\ \quad -\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}+\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\ \quad -\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)},\end{array}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta+\varepsilon}$ divisa dat hanc aequationem maxime notatu dignam:

$$\left.\begin{array}{l}\dfrac{1}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)}\\ \quad-\dfrac{1}{\delta\varepsilon(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}+\dfrac{1}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\ \quad-\dfrac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}+\dfrac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\ \quad-\dfrac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{array}\right\}=0.$$

19. Haec theoremata eo magis sunt memorabilia, quod eorum veritas non nisi per plures ambages in numeris explorari potest, ideoque multo maiorem attentionem merentur, quam aliud simile theorema, ad quod nuper sum perductus, quippe cuius demonstratio haud difficulter exhiberi potest, quod ita se habet.

THEOREMA NUMERICUM

Sumtis pro lubitu quotcunque numeris veluti quatuor α, β, γ, δ, si hinc totidem alii sequenti modo formentur

$$a=\alpha,\ b=\alpha+\beta,\ c=\alpha+\beta+\gamma \quad \text{et} \quad d=\alpha+\beta+\gamma+\delta,$$

similique modo etiam isti

$$D=\delta,\ C=\delta+\gamma,\ B=\delta+\gamma+\beta \quad \text{et} \quad A=\delta+\gamma+\beta+\alpha,$$

tum semper erit

$$\frac{1}{abcd}-\frac{1}{abcD}+\frac{1}{abCD}-\frac{1}{aBCD}+\frac{1}{ABCD}=0.$$

DEMONSTRATIO

20. Binae fractiones priores inventae, ob $D-d=-c$, dant fractionem $-\frac{1}{abdD}$, quae cum tertia coniuncta producit $\frac{1}{adCD}$, cui quarta fractio iuncta dat $-\frac{1}{dBCD}$, quae (ob $d=A$) a termino ultimo penitus destruitur.

21. Ope superiorum theorematum omnes formulae integrales implicatae, ad quas integratio aequationum linearum perducere solet, facile resolvi poterunt. Pervenitur autem plerumque ad tales formas:

$$Z = \int \partial q \int X \partial p, \quad Z = \int \partial r \int \partial q \int X \partial p,$$
$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p, \quad Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p \quad \text{etc.},$$

ubi litterae p, q, r, s, t etc. sunt functiones datae ipsius X, at vero X functio quaecunque ipsius x; atque hic tota resolutio ita institui debet, ut in singulis membris functio haec indefinita X unicum tantum signum integrale habeat praefixum; hoc igitur, ope superiorum theorematum, facile praestari poterit, si modo ibi loco elementi ∂p scribamus $X \partial p$, quo observato singulae reductiones sequenti modo se habebunt.

I. RESOLUTIO FORMULAE INTEGRALIS $\int \partial q \int X \partial p$

22. Si loco ∂p scribamus $X \partial p$ theorema secundum § 11 nobis suppeditat hanc aequationem:

$$\int X \partial p \int \partial q - \int X \partial p \cdot \int \partial q + \int \partial q \int X \partial p = 0,$$

cuius postremum membrum est ipsa nostra forma reducenda Z, consequenter resolutio statim dat

$$Z = \int \partial q \cdot \int X \partial p - \int X \partial p \int \partial q,$$

ideoque ob $\int \partial q = q$ habebimus

$$Z = q \int X \partial p - \int X q \partial p.$$

COROLLARIUM

23. Si fuerit $q = p$, erit

$$Z = p \int X \partial p - \int X p \partial p.$$

II. RESOLUTIO FORMULAE IMPLICATAE $\int \partial r \int \partial q \int X \partial p$

24. Pro hoc casu sumamus theorema 3 § 12, unde, si loco ∂p scribatur $X \partial p$, deducimus hanc aequationem:

$$\int X \partial p \int \partial q \int \partial r - \int X \partial p \int \partial q \cdot \int \partial r + \int X \partial p \cdot \int \partial r \int \partial q - \int \partial r \int \partial q \int X \partial p = 0,$$

cuius postremum membrum est ipsa forma reducenda Z, hincque adeoque colligitur

$$Z = \int \partial r \int \partial q \cdot \int X \partial p - \int \partial r \cdot \int X \partial p \int \partial q + \int X \partial p \int \partial q \int \partial r,$$

quae ergo reducta dat

$$Z = \int q \partial r \cdot \int X \partial p - r \int X q \partial p + \int X \partial p \int r \partial q.$$

COROLLARIUM

25. Si ergo hic fuerit $q = r = p$, prodibit ista resolutio:

$$Z = \int \partial p \int \partial p \int X \partial p = \tfrac{1}{2} pp \int X \partial p - p \int X p \partial p + \tfrac{1}{2} \int X pp \partial p.$$

III. RESOLUTIO HUIUS FORMULAE IMPLICATAE $Z = \int \partial s \int \partial r \int \partial q \int X \partial p$.

26. Pro hoc casu sumamus theorema 4 § 13, unde, si loco ∂p scribatur $X \partial p$, deducimus hanc aequationem:

$$\left.\begin{aligned} \int X \partial p \int \partial q \int \partial r \int \partial s - \int X \partial p \int \partial q \int \partial r \cdot \int \partial s + \int X \partial p \int \partial q \cdot \int \partial s \int \partial r \\ - \int X \partial p \cdot \int \partial s \int \partial r \int \partial q + \int \partial s \int \partial r \int \partial q \int X \partial p \end{aligned}\right\} = 0,$$

cuius postremum membrum est ipsa nostra formula reducenda Z; hincque adeo colligimus

$$Z = \int \partial s \int \partial r \int \partial q \cdot \int X \partial p - \int \partial s \int \partial r \cdot \int X \partial p \int \partial q + \int \partial s \cdot \int X \partial p \int \partial q \int \partial r - \int X \partial p \int \partial q \int \partial r \int \partial s,$$

quae ergo reducta praebet

$$Z = \int \partial s \int q \partial r \cdot \int X \partial p - \int r \partial s \cdot \int X q \partial p + s \int X \partial p \int r \partial q - \int X \partial p \int \partial q \int s \partial r.$$

COROLLARIUM

27. Si ponatur $s = r = q = p$, tum prodibit ista resolutio

$$Z = \tfrac{1}{6} p^3 \int X \partial p - \tfrac{1}{2} pp \int X p \partial p + \tfrac{1}{2} p \int X pp \partial p - \tfrac{1}{6} \int X p^3 \partial p.$$

IV. RESOLUTIO HUIUS FORMULAE IMPLICATAE $Z = \int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p$

28. Pro hoc casu sumamus theorema 5 § 14, unde, si loco ∂p scribatur $X\partial p$, prodibit ista aequatio:

$$\left.\begin{matrix}\int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t - \int X\partial p\int\partial q\int\partial r\int\partial s\cdot\int\partial t + \int X\partial p\int\partial q\int\partial r\cdot\int\partial t\int\partial s\\ - \int X\partial p\int\partial q\cdot\int\partial t\int\partial s\int\partial r + \int X\partial p\cdot\int\partial t\int\partial s\int\partial r\int\partial q - \int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p\end{matrix}\right\} = 0,$$

cuius postremum membrum est ipsa nostra forma reducenda Z, unde ergo prodit

$$Z = \left\{\begin{matrix}\int\partial t\int\partial s\int\partial r\int\partial q\cdot\int X\partial p - \int\partial t\int\partial s\int\partial r\cdot\int X\partial p\int\partial q + \int\partial t\int\partial s\cdot\int X\partial p\int\partial q\int\partial r\\ - \int\partial t\cdot\int X\partial p\int\partial q\int\partial r\int\partial s + \int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t,\end{matrix}\right.$$

quae ergo reducta praebet

$$Z = \left\{\begin{matrix}\int\partial t\int\partial s\int q\partial r\cdot\int X\partial p - \int\partial t\int r\partial s\cdot\int Xq\partial p + \int s\partial t\cdot\int X\partial p\int r\partial q\\ - t\int X\partial p\int\partial q\int s\partial r + \int X\partial p\int\partial q\int\partial r\int t\partial s.\end{matrix}\right.$$

COROLLARIUM

29. Si hic sumatur $t = s = r = q = p$, tum prodibit ista resolutio

$$Z = \tfrac{1}{24}p^4\int X\partial p - \tfrac{1}{6}p^3\int Xp\partial p + \tfrac{1}{4}pp\int Xpp\partial p - \tfrac{1}{6}p\int Xp^3\partial p + \tfrac{1}{24}\int Xp^4\partial p.$$

30. Quo indoles harum resolutionum clarius perspiciatur, quoniam litterae p, q, r, s, t functiones datas ipsius x denotant, ideoque omnes expressiones ex iis formatae pariter ut cognitae spectari possunt, statuamus brevitatis gratia

$$\partial p\int\partial q = \partial p', \quad \partial p\int\partial q\int\partial r = \partial p'', \quad \partial p\int\partial q\int\partial r\int\partial s = \partial p''',$$
$$\partial p\int\partial q\int\partial r\int\partial s\int\partial t = \partial p^{IV} \quad \text{etc.},$$

hocque modo postrema resolutio ita referetur:

$$Z = \int\partial t\int\partial s\int\partial r\int\partial q\cdot\int X\partial p - \int\partial t\int\partial s\int\partial r\cdot\int X\partial p'$$
$$+ \int\partial t\int\partial s\cdot\int X\partial p'' - \int\partial t\cdot\int X\partial p''' + \int X\partial p^{IV}.$$

Quod si hic porro statuamus

$$\int\partial t\int\partial s = \int s\partial t = t', \quad \int\partial t\int\partial s\int\partial r = t'', \quad \int\partial t\int\partial s\int\partial r\int\partial q = t''',$$

tota resolutio hoc modo concinne repraesentabitur

$$Z = t''' \int X\partial p - t'' \int X\partial p' + t' \int X\partial p'' - t\int X\partial p''' + \int X\partial p^{IV},$$

quam repraesentationem etiam ad praecedentes resolutiones accommodasse iuvabit.

31. Cum igitur integratio formulae implicatae

$$Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X\partial p$$

reducatur ad integrationem sequentium formularum integralium simplicium:

$$\int X\partial p,\ \int X\partial p',\ \int X\partial p'',\ \int X\partial p''',\ \int X\partial p^{IV},$$

quaestio hinc oritur non parum curiosa, quemadmodum ex his formulis simplicibus vicissim quantitates q, r, s et t concludi queant, quod sequenti modo facile praestabitur. Cum sit $\partial p' = \partial p \int \partial q$, erit $\int \partial q = q = \frac{\partial p'}{\partial p}$. Ponatur nunc porro

$$\frac{\partial p''}{\partial p} = q',\ \frac{\partial p'''}{\partial p} = q'',\ \frac{\partial p^{IV}}{\partial p} = q''' \text{ etc.},$$

quibus valoribus introductis habebimus

$$q' = \int \partial q \int \partial r,\ \ q'' = \int \partial q \int \partial r \int \partial s,\ \ q''' = \int \partial q \int \partial r \int \partial s \int \partial t \quad \text{etc.}$$

Quoniam igitur hi valores q, q', q'', q''', q^{IV} sunt dati, ex prima statim colligimus $\int \partial r = \frac{\partial q'}{\partial q} = r$. Ponamus autem porro $\frac{\partial q''}{\partial q} = r'$, $\frac{\partial q'''}{\partial q} = r''$ etc. eruntque etiam hi valores, r, r', r'' etc. dati, quibus substitutis habebitur

$$r' = \int \partial r \int \partial s,\ \ r'' = \int \partial r \int \partial s \int \partial t,$$

ex quarum prima sequitur $\int \partial s = s = \frac{\partial r'}{\partial r}$. Quare si porro fiat $s' = \frac{\partial r''}{\partial r}$, erit quoque $s' = \int \partial s \int \partial t$, hincque $\int \partial t = t = \frac{\partial s'}{\partial s}$. Ex his clare intelligitur, quomodo hae formulae inveniri queant pro casibus adhuc magis complicatis.

32. Superest, ut etiam de transformatione talium formularum integralium implicatarum pauca adiiciamus, quod totum negotium sequenti problemate includi potest.

PROBLEMA

33. *Proposita formula implicata terna signa summatoria involvente* $\int\partial p\int\partial q\int\partial r$, *investigare aliam similem formulam*

$$\int\partial P\int\partial Q\int\partial R,$$

illi aequalem[1]).

SOLUTIO

Per theorema 2 supra allatum formula proposita ita est resoluta:

$$\int\partial q\int\partial r=\int\partial q\cdot\int\partial r-\int\partial r\int\partial q=q\int\partial r-\int q\partial r.$$

Simili modo pro formula quaesita erit

$$\int\partial Q\int\partial R=Q\int\partial R-\int Q\,\partial R;$$

requiritur igitur, ut sit

$$q\,\partial p\int\partial r-\partial p\int q\partial r=Q\,\partial P\int\partial R-\partial P\int Q\,\partial R,$$

quae aequalitas adimpleretur, sumendo $P=p$, $Q=q$ et $R=r$; verum permutandis membris statuamus

$$Q\,\partial P\int\partial R=-\,\partial p\int q\,\partial r\quad\text{et}\quad\partial P\int Q\,\partial R=-\,q\,\partial p\int\partial r,$$

atque ex priore aequatione deducimus

$$Q\partial P=-\,\partial p,\quad\text{ideoque}\quad\partial P=-\frac{\partial p}{Q},$$

tum vero

$$\partial R=q\,\partial r;$$

ex altera vero aequatione habemus

$$\partial P=-\,q\,\partial p\quad\text{et}\quad Q\,\partial R=\partial r.$$

Cum igitur esset $\partial P=-\frac{\partial p}{Q}$, erit $Q=\frac{1}{q}$, hincque porro $\partial R=q\,\partial r$, unde ob $Q=\frac{1}{q}$, erit $\partial Q=\frac{-\partial q}{qq}$. Consequenter formula integralis quaesita proposita $\int\partial p\int\partial q\int\partial r$ aequalis erit

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1) Nisi functiones p, q, r peculiari modo erunt determinatae, oportebit esse

$$dP=aq\,dp,\qquad dQ=\frac{dq}{acq^2},\qquad dR=cq\,dr,$$

designantibus a et c constantes arbitrarias. H. D.

$$\int q\partial p\int\frac{\partial q}{qq}\int q\partial r,$$

unde patet perpetuo loco formulae $\int\partial p\int\partial q\int\partial r$ scribi posse istam:

$$\int q\partial p\int\frac{\partial q}{qq}\int q\partial r.$$

COROLLARIUM 1

34. Quando igitur plura signa integralia sibi invicem fuerint involuta, veluti si habeamus $\int\partial p\int\partial q\int\partial r\int\partial s$, ista transformatio in quibusvis ternis signis se mutuo sequentibus institui poterit, unde in hac formula proposita duplex transformatio adhiberi poterit; prior scilicet in ternis signis prioribus praebebit

$$\int q\partial p\int\frac{\partial q}{qq}\int q\partial r\int\partial s,$$

at vero in ternis posterioribus haec transformatio adhibita dabit

$$\int\partial p\int r\partial q\int\frac{\partial r}{rr}\int r\partial s.$$

COROLLARIUM 2

35. Hinc porro ope eiusdem transformationis aliae insuper fieri possunt, veluti ex postrema forma

$$\int\partial p\int r\partial q\int\frac{\partial r}{rr}\int r\partial s,$$

ut in ternis prioribus signis res expediri queat, loco $r\,\partial q$ scribamus ∂v, ut habeamus

$$\int\partial p\int\partial v\int\frac{\partial r}{rr}\int r\partial s,$$

quae transformatur in hanc:

$$\int v\partial p\int\frac{\partial v}{vv}\int\frac{v\partial r}{rr}\int r\partial s,$$

quae omnes formulae ipsi propositae sunt prorsus aequales.

36. Ut rem exemplo illustremus, sumamus esse $p = x^\alpha$, $q = x^\beta$, $r = x^\gamma$, ita ut formula proposita sit

$$\alpha\beta\gamma \int x^{\alpha-1}\partial x \int x^{\beta-1}\partial x \int x^{\gamma-1}\partial x = \frac{\alpha\beta x^{\gamma+\beta+\alpha}}{(\gamma+\beta)(\gamma+\beta+\alpha)}.$$

Iam pro transformatione erit primo

$$\int q\,\partial r = \frac{\gamma x^{\beta+\gamma}}{\beta+\gamma}, \quad \text{ideoque ob} \quad \frac{\partial q}{qq} = \frac{\beta\,\partial x}{x^{\beta+1}}$$

erit

$$\int \frac{\partial q}{qq} \int q\,\partial r = \frac{\beta x^\gamma}{\beta+\gamma},$$

quod ductum in $q\,\partial p$ et integratum producit

$$\frac{\alpha\beta x^{\alpha+\beta+\gamma}}{(\beta+\gamma)(\alpha+\beta+\gamma)}.$$

Patet igitur hanc transformationem latissime patere, atque ad omnes formulas implicatas accommodari posse eo pluribus diversis modis, quo plura signa integralia invicem involvantur.

37. Haud abs re fore iudico resolutiones supra traditas ad summationem serierum potestatum reciprocarum applicare, quod fiet si loco X sumamus fractionem $\frac{x}{1-x}$, tum vero pro singulis elementis ∂p, ∂q, ∂r, ∂s scribamus $\frac{\partial x}{x}$, unde corollaria subnexa in usum vocari poterunt, ubi scilicet erit $p = lx$.

38. Cum sit per seriem infinitam

$$X = x + xx + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.},$$

erit

$$\int X\partial p = \int \frac{X\partial x}{x} = x + \tfrac{1}{2}xx + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \tfrac{1}{5}x^5 + \tfrac{1}{6}x^6 + \text{etc.},$$

quam seriem constat exprimere logarithmum fractionis $\frac{1}{1-x}$, quandoquidem est

$$\int \frac{X\partial x}{x} = -l(1-x) = l\frac{1}{1-x}.$$

39. Multiplicetur haec series porro per $\frac{\partial x}{x}$ et integretur, prodibitque

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=x+\frac{1}{4}xx+\frac{1}{9}x^3+\frac{1}{16}x^4+\frac{1}{25}x^5+\text{ etc.},$$

at vero huius formulae integralis resolutio supra § 22 data praebet

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=lx\int\frac{\partial x}{1-x}-\int\frac{\partial x\, lx}{1-x},$$

quae quidem integralia ita accipi supponuntur, ut posito $x=0$ evanescant; hic autem imprimis notetur, casu quo sumitur $x=1$, ob $l1=0$, huius seriei:

$$1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+\text{ etc.}$$

summam fore $-\int\frac{\partial x\, lx}{1-x}$, cuius valorem olim primus inveni esse $=\frac{\pi\pi}{6}$.

40. Ducamus superiorem seriem denuo in $\frac{\partial x}{x}$ et integrando obtinebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^3}xx+\frac{1}{3^3}x^3+\frac{1}{4^3}x^4+\frac{1}{5^3}x^5+\text{ etc.}$$

Formula autem haec implicata per § 25 ita resolvitur:

$$\frac{1}{2}(lx)^2\int\frac{\partial x}{1-x}-lx\int\frac{\partial x\, lx}{1-x}+\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}.$$

Casu igitur, quo $x=1$, summa seriei reciprocae cuborum

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{ etc.}$$

erit $=\frac{1}{2}\int\frac{\partial x(lx)^2}{1-x}$.

41. Simili modo superiorem seriem per $\frac{\partial x}{x}$ multiplicemus et integremus, tum prodibit

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\frac{1}{2^4}xx+\frac{1}{3^4}x^3+\frac{1}{4^4}x^4+\text{ etc.}$$

At vero haec formula implicata per § 27 reducitur ad hanc formam:

$$\frac{1}{6}(lx)^3\int\frac{\partial x}{1-x}-\frac{1}{2}(lx)^2\int\frac{\partial x\, lx}{1-x}+\frac{1}{2}lx\int\frac{\partial x(lx)^2}{1-x}-\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x}.$$

Pro casu ergo quo $x = 1$ huius seriei reciprocae biquadratorum summa erit $-\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x}$, cuius valorem olim ostendi esse $\frac{\pi^4}{90}$.

42. Multiplicatione denuo per $\frac{\partial x}{x}$ instituta et integratione peracta habebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x} = x + \frac{1}{2^5}xx + \frac{1}{3^5}x^3 + \frac{1}{4^5}x^4 + \frac{1}{5^5}x^5 + \text{etc.},$$

quae formula implicata per § 29 reducitur ad hanc formam:

$$\frac{1}{24}(lx)^4\int\frac{\partial x}{x} - \frac{1}{6}(lx)^3\int\frac{\partial x\, lx}{1-x} + \frac{1}{4}(lx)^2\int\frac{\partial x(lx)^2}{1-x} - \frac{1}{6}lx\int\frac{\partial x(lx)^3}{1-x} + \frac{1}{24}\int\frac{\partial x(lx)^4}{1-x}.$$

Hinc ergo casu $x = 1$ huius seriei reciprocae potestatum quintarum summa erit $\frac{1}{24}\int\frac{\partial x(lx)^4}{1-x}$.

43. Colligamus omnes istas series pro casu $x = 1$, earumque summae sequenti modo per formulam integralem simplicem exprimentur:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} = \int\frac{\partial x}{1-x} = \infty,$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = -\int\frac{\partial x\, lx}{1-x} = \frac{\pi\pi}{6},$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \frac{1}{2}\int\frac{\partial x(lx)^2}{1-x},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = -\frac{1}{6}\int\frac{\partial x(lx)^3}{1-x} = \frac{\pi^4}{90},$$

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \frac{1}{24}\int\frac{\partial x(lx)^4}{1-x},$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = -\frac{1}{120}\int\frac{\partial x(lx)^5}{1-x} = \frac{\pi^6}{945},$$

$$1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \frac{1}{5^7} + \text{etc.} = \frac{1}{720}\int\frac{\partial x(lx)^6}{1-x},$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = -\frac{1}{5040}\int\frac{\partial x(lx)^7}{1-x} = \frac{\pi^8}{9450}$$

etc.

44. In genere igitur huius seriei:

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{etc.}$$

in infinitum continuatae summa ita exprimetur:

$$\pm\frac{1}{1.2.3\ldots.(n-1)}\int\frac{\partial x(lx)^{n-1}}{1-x},$$

ubi signum superius $+$ valet, quando exponens n est impar, inferius vero, quando est par. Istas summationes, iam pridem[1]) quidem repertas, ideo hic afferre visum est, quod non ita pridem Celeberr. LORGNA[2]) easdem has summationes per formulas continuo magis implicatas expressas exhibuit, cum sine dubio istae formulae integrales simplices longe praeferendae videantur.

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1) Vide L. EULERI Commentationes 393, 463, 464 (indicis *Enestroemiani*). *De summis serierum numeros* BERNOULLIANOS *involventium. De valore formulae integralis*

$$\int\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\frac{dz}{z}(lz)^{\mu}$$

casu, quo post integrationem ponitur $z=1$. *Nova methodus quantitates integrales determinandi.* Novi Comment. acad. sc. Petrop. 14 I, 1770, p. 129; 19, 1775, p. 30; 19, 1775, p. 66, § 41—47. Vide quoque *Introductionis in analysin infinitorum* t. I, § 168 et *Institutionum calculi differentialis* partem posteriorem § 151. *LEONHARDI EULERI Opera omnia*, series I, vol. 14, 17, 10. H. D.

2) ANTONIO MARIA LORGNA (1735—1796). H. D.

DE AEQUATIONIBUS DIFFERENTIALIBUS CUIUSCUNQUE GRADUS QUAE DENUO DIFFERENTIATAE INTEGRARI POSSUNT

M. S. Academiae exhibuit die 8. Octobris 1781

Commentatio 680 indicis ENESTROEMIANI

Institutiones calculi integralis 4, 1794, p. 564—577

1. Sint x et y binae variabiles, inter quas earumque differentialia cuiuscunque gradus aequationes propositae subsistant. Ad formam differentialium tollendam ponatur more solito

$$\partial y = p\,\partial x,\ \ \partial p = q\,\partial x,\ \ \partial q = r\,\partial x,\ \ \partial r = s\,\partial x\ \ \text{etc.},$$

ita ut, sumto elemento ∂x constante, sit

$$p = \frac{\partial y}{\partial x},\ \ q = \frac{\partial\partial y}{\partial x^2},\ \ r = \frac{\partial^3 y}{\partial x^3},\ \ s = \frac{\partial^4 y}{\partial x^4}\ \ \text{etc.}$$

Sint porro P et $\mathfrak{P}$ functiones quaecunque ipsius p, Q et $\mathfrak{Q}$ functiones quaecunque ipsius q, R et $\mathfrak{R}$ ipsius r, S et $\mathfrak{S}$ ipsius s etc., quae functiones non solum esse possunt rationales, sed etiam irrationales, atque adeo transcendentes.

2. His positis duo aequationum genera per differentiationem integrare docebo, quarum primum istas continet aequationes:

$$y - px = P,\ \ p - qx = Q,\ \ q - rx = R,\ \ r - sx = S\ \ \text{etc.},$$

quarum prima involvere potest functiones quascunque ipsius ∂y, tam rationales quam irrationales, quin etiam functiones transcendentes; secunda tales functiones ipsius $\partial\partial y$ involvere potest, tertia ipsius $\partial^3 y$, quarta ipsius $\partial^4 y$, et ita porro, cuiusmodi aequationum integratio certe nemini adhuc in mentem venire potuit.

3. Alterum genus aequationum, quarum integrationem per differentiationem expedire docebo, sequentes complectitur aequationes:

$$y + \mathfrak{P}x = P, \quad p + \mathfrak{Q}x = Q, \quad q + \mathfrak{R}x = R, \quad r + \mathfrak{S}x = S \quad \text{etc.},$$

quae duas functiones quascunque involvunt. Evidens autem est has aequationes praecedentes in se comprehendere, quando scilicet est

$$\mathfrak{P} = -p, \quad \mathfrak{Q} = -q, \quad \mathfrak{R} = -r, \quad \mathfrak{S} = -s \quad \text{etc.}$$

Ceterum patet, has aequationes adeo complicatas esse posse, ut nemo certe earum integrationem suscipere voluerit.

DE AEQUATIONIBUS PRIORIS GENERIS

PROBLEMA 1

4. *Proposita aequatione differentiali primi gradus $y - px = P$, eius integrale completum invenire*[1]).

SOLUTIO

Cum sit $\partial y = p\,\partial x$, si aequatio proposita differentietur, prodibit haec $-x\,\partial p = \partial P$, unde, posito $\partial P = P'\,\partial p$, colligitur $x = -P'$. Quod si iam p tanquam novam variabilem spectemus, per eam tam x quam y exprimere poterimus. Cum enim sit $y = px + P$, erit $y = P - pP'$, unde, eliminando p, quoties quidem calculus id permittet, conflari poterit aequatio inter x et y, quae autem tantum ut integrale particulare spectari debet, quia nullam involvit constantem arbitrariam. At vero, quoniam aequationem per differen-

1) Cf. L. Euleri Commentationem 236 (indicis *Enestroemiani*). *Exposition de quelques paradoxes dans le calcul intégral.* Mém. Berlin 12, 1758, p. 304; *Institutiones calculi integralis* vol. I, § 695—703. *Leonhardi Euleri Opera omnia*, series I, vol. 22, 12. H. D.

tiationem erutam $-x\partial p = P'\partial p$ per ∂p dividere licuit, iste factor nihilo aequatus integrale completum suppeditare est censendus. Posito enim $\partial p = 0$ erit $p = \text{const.} = \alpha$, ideoque $y = \int p\,\partial x = \alpha x + \beta$. Haec quidem aequatio duas constantes arbitrarias involvere videtur; at vero altera per ipsam aequationem propositam determinatur, cum facta substitutione fiat

$$\alpha x + \beta - \alpha x = P, \quad \text{ideoque} \quad \beta = P = f:\alpha.$$

PROBLEMA 2

5. *Proposita aequatione differentiali secundi gradus $p - qx = Q$, eius integrale completum assignare.*

SOLUTIO

Si haec aequatio differentietur et loco ∂p scribatur $q\,\partial x$, prodibit ista $-x\partial q = \partial Q$, sive, posito $\partial Q = Q'\,\partial q$, erit $-x\partial q = Q'\,\partial q$. Hinc factor communis ∂q nihilo aequatus praebet $q = \text{const.} = 2\alpha$, unde fit

$$p = \int q\,\partial x = 2\alpha x + \beta, \quad \text{hincque}$$
$$y = \int p\,\partial x = \alpha xx + \beta x + \gamma,$$

quarum trium constantium α, β, γ una per aequationem propositam determinatur. Facta autem divisione per ∂q habebimus $x = -Q'$, unde colligitur

$$p = Q + qx = Q - qQ',$$

hincque ob $\partial x = -\partial Q' = -Q''\,\partial q$, erit

$$y = \int p\,\partial x = \int Q''\,\partial q(Q'q - Q) + b.$$

EXEMPLUM

6. Sit $Q = aq^m$, erit

$$Q' = maq^{m-1}$$

atque

$$Q'' = m(m-1)aq^{m-2}.$$

Hoc ergo casu erit

$$x = -Q' = -maq^{m-1},$$

$$y = m(m-1)^2 aa \int q^{2m-2} \partial q + b,$$

sive

$$y = \frac{m(m-1)^2}{2m-1} aaq^{2m-1} + b.^{1)}$$

Est vero $q^{m-1} = -\frac{x}{ma}$, ita ut valor ipsius y facile per x exprimi poterit, quo facto habebitur integrale completum huius aequationis differentio-differentialis

$$\frac{\partial y}{\partial x} - \frac{x \partial\partial y}{\partial x^2} = \frac{a(\partial\partial y)^m}{\partial x^{2m}}.$$

PROBLEMA 3

7. *Proposita aequatione differentiali tertii gradus $q - rx = R$, eius integrale completum investigare.*

SOLUTIO

Haec aequatio differentiata, ob $\partial q = r\partial x$, dat $-x\partial r = \partial R = R'\partial r$, cuius aequationis factor ∂r nihilo aequatus hanc suppeditabit aequationem:

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

ubi quatuor constantium α, β, γ, δ una ex ipsa aequatione proposita determinata habebitur. Cum enim hinc sit

$$p = 3\alpha x^2 + 2\beta x + \gamma, \quad q = 6\alpha x + 2\beta, \quad r = 6\alpha,$$

erit substituendo $2\beta = R$, ita ut tres tantum constantes arbitrariae in calculo relinquantur, uti natura huiusmodi aequationum postulat. Facta autem divisione per ∂r satisfaciet aequatio $x = -R'$, unde colligitur $q = R - rR'$. Hinc, ob

1) Editio princeps: $\frac{m(m-1)}{2}$ loco $\frac{m(m-1)^2}{2m-1}$. Correxit H. D.

$$\partial x = -\partial R' = -R''\partial r,$$

reperietur

$$p = \int q\,\partial x = \int R''\partial r(rR' - R),$$

ac denique $y = \int p\,\partial x$, ubi ob duplicem integrationem duae constantes arbitrariae inseruntur.

EXEMPLUM

8. Sit $R = ar^m$, erit

$$R' = mar^{m-1} \quad \text{et} \quad R'' = m(m-1)ar^{m-2},$$

unde colligitur

$$p = \frac{m(m-1)^2}{2m-1}aar^{2m-1} + b\,,^{1)}$$

atque ob

$$\partial x = -\partial R' = -R''\partial r = -m(m-1)ar^{m-2}\partial r,$$

nanciscimur

$$y = \int p\partial x = -\frac{m^2(m-1)^3}{(2m-1)(3m-2)}a^3r^{3m-2} - mabr^{m-1} + c,^{2)}$$

unde ob $r^{m-1} = -\frac{x}{ma}$ facile obtinetur aequatio finita inter x et y, haecque erit integrale completum huius aequationis differentialis tertii gradus:

$$\frac{\partial\partial y}{\partial x^2} - \frac{x\partial^3 y}{\partial x^3} = \frac{a(\partial^3 y)^m}{\partial x^{3m}}.$$

PROBLEMA 4

9. *Proposita aequatione differentiali quarti gradus $r - sx = S$, eius integrale completum indagare.*

SOLUTIO

Ob $\partial r = s\partial x$ fiet, aequationem propositam differentiando, $-x\partial s = \partial S = S'\partial s$, cuius aequationis factor ∂s praebet aequationem finitam

1) Vide notam p. 271. H. D.

2) Editio princeps: $y = \int p\,\partial x = -\frac{m^2(m-1)}{2\cdot 3}a^3r^{3m-2} - \frac{m(m-1)}{m-2}abr^{m-1} + c$. Correxit H. D.

$$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon,$$

ubi una constantium per ipsam aequationem propositam determinatur. Porro satisfacit aequatio $x = -S'$, unde colligitur $r = S - sS'$, hincque, ob

$$\partial x = -\partial S' = -S''\partial s$$

reperitur

$$q = \int r\partial x, \quad p = \int q\partial x \quad \text{et} \quad y = \int p\partial x,$$

sive

$$y = \int \partial x \int \partial x \int r\partial x,$$

ubi ob triplicem integrationem tres adiiciendae sunt constantes arbitrariae. Simili modo ad aequationes altiorum graduum progredi licet.

DE AEQUATIONIBUS SECUNDI GENERIS

PROBLEMA 5

10. *Proposita aequatione differentiali primi gradus huiusmodi* $y + \mathfrak{P}x = P$, *eius integrale completum investigare.*

SOLUTIO

Si ista aequatio $y + \mathfrak{P}x = P$ differentietur, et loco ∂y scribatur $p\partial x$, prodit haec:

$$p\partial x + \mathfrak{P}\partial x + x\partial\mathfrak{P} = \partial P,$$

sive posito $\partial P = P'\partial p$, erit

$$(p + \mathfrak{P})\partial x + x\partial\mathfrak{P} = P'\partial p,$$

quae per $p + \mathfrak{P}$ divisa dat

$$\partial x + x \cdot \partial \cdot l(p + \mathfrak{P}) - \frac{x\partial p}{p + \mathfrak{P}} = \frac{P'\partial p}{p + \mathfrak{P}}.$$

$$\left(\text{Est enim } \frac{x\partial\mathfrak{P}}{p + \mathfrak{P}} = x \cdot \partial \cdot l(p + \mathfrak{P}) - \frac{x\partial p}{p + \mathfrak{P}}\right).$$

Quod si iam ponamus

$$\int \frac{\partial p}{p + \mathfrak{P}} = z,$$

aequatio illa integrabilis reddetur multiplicando per $e^{-z}(p+\mathfrak{P})$. Prodit enim

$$(p+\mathfrak{P})e^{-z}\partial x+(p+\mathfrak{P})xe^{-z}\partial\cdot l(p+\mathfrak{P})-x\partial z e^{-z}(p+\mathfrak{P})=e^{-z}P'\partial p,$$

cuius integrale manifesto est

$$xe^{-z}(p+\mathfrak{P})=\int e^{-z}P'\partial p,$$

unde colligitur

$$x=\frac{e^z}{p+\mathfrak{P}}\int e^{-z}P'\partial p=\frac{e^z}{p+\mathfrak{P}}\int e^{-z}\partial P,$$

unde statim fit

$$y=P-\frac{\mathfrak{P}e^z}{p+\mathfrak{P}}\int e^{-z}\partial P,$$

ubi e^z est etiam functio ipsius p, ita ut ambae variabiles x et y per unam eandemque variabilem p exprimantur, quae expressiones iam constantem arbitrariam per se involvunt, ita ut eius adiectione non amplius opus sit.

EXEMPLUM

11. Sit $P=ap^m$ et $\mathfrak{P}=bp^n$, ita ut aequatio integranda sit $y+bp^nx=ap^m$. Hic igitur erit

$$z=\int\frac{\partial p}{p(1+bp^{n-1})}=\int\frac{\partial p}{p}-b\int\frac{p^{n-2}\partial p}{1+bp^{n-1}},$$

unde colligitur actu integrando

$$z=lp-\frac{1}{n-1}l(1+bp^{n-1}),$$

ex quo fit

$$e^z=\frac{p}{(1+bp^{n-1})^{\frac{1}{n-1}}}\quad\text{et}\quad e^{-z}=\frac{(1+bp^{n-1})^{\frac{1}{n-1}}}{p},$$

quamobrem habebimus

$$\int e^{-z}\partial P=am\int p^{m-2}(1+bp^{n-1})^{\frac{1}{n-1}}\partial p,$$

in qua expressione nullae quantitates transcendentes insunt, ita ut x et y facile definiantur, hocque modo obtinetur integrale completum istius aequationis differentialis primi gradus

$$y+bx\frac{\partial y^n}{\partial x^n}=a\frac{\partial y^m}{\partial x^m}.$$

PROBLEMA 6

12. *Proposita hac aequatione differentiali secundi gradus:* $p + \mathfrak{Q}x = Q$, *eius integrale completum invenire.*

SOLUTIO

Attendenti mox patebit, hanc aequationem ex praecedente oriri, si loco y, P, $\mathfrak{P}$, scribantur litterae p, Q, $\mathfrak{Q}$, quandoquidem litterae y, p, q, r etc. uniformi lege progrediuntur; quamobrem facta hac immutatione ex praecedente solutione statim habebimus

$$x = \frac{e^z}{q + \mathfrak{Q}} \int e^{-z} \partial Q, \quad \text{existente} \quad z = \int \frac{\partial q}{q + \mathfrak{Q}};$$

sicque x hic erit functio solius quantitatis q, ex qua fit

$$\partial x = \frac{Q' - x\mathfrak{Q}'}{q + \mathfrak{Q}} \partial q.$$

Deinde nunc etiam p per solam variabilem q definietur: erit enim per § 10

$$p = Q - \frac{\mathfrak{Q} e^z}{q + \mathfrak{Q}} \int e^{-z} \partial Q.$$

Cum igitur sit $y = \int p \partial x$, etiam quantitas y per solam functionem ipsius q exprimetur, hocque modo problema perfecte solutum est censendum.

PROBLEMA 7

13. *Proposita aequatione differentiali tertii gradus hac:* $q + \mathfrak{R}x = R$, *eius integrale completum assignare.*

SOLUTIO

Haec solutio simili modo ex problemate primo huius secundi generis (§ 10) derivari potest, dum loco y, P, $\mathfrak{P}$, scribatur q, R, $\mathfrak{R}$, id quod si primo in aequatione pro x fuerit factum, suppeditabit hanc expressionem:

$$x = \frac{e^z}{r + \mathfrak{R}} \int e^{-z} \partial R, \quad \text{existente} \quad z = \int \frac{\partial r}{r + \mathfrak{R}},$$

sicque x erit functio solius variabilis r; tum vero erit

$$\partial x = \frac{R' - x\mathfrak{R}'}{r + \mathfrak{R}} \partial r.$$

Formula porro ibi pro y inventa et huc translata dabit pro q hanc expressionem:

$$q = R - \frac{\mathfrak{R} e^z}{r + \mathfrak{R}} \int e^{-z} \partial R,$$

quae etiam tantum variabilem r eiusque functiones involvit. Quia igitur $p = \int q \, \partial x$ et $y = \int p \, \partial x$, erit $y = \int \partial x \int q \, \partial x$, sicque etiam y per solam variabilem r exprimetur.

PROBLEMA 8

14. *Proposita aequatione differentiali quarti gradus $r + \mathfrak{S} x = S$, eius integrale investigare.*

SOLUTIO

Hic erit

$$x = \frac{e^z}{s + \mathfrak{S}} \int e^{-z} \partial S, \quad \text{existente} \quad z = \int \frac{\partial s}{s + \mathfrak{S}}.$$

Porro erit

$$\partial x = \frac{S' - x\mathfrak{S}'}{s + \mathfrak{S}} \partial s, \quad r = S - \frac{\mathfrak{S} e^z}{s + \mathfrak{S}} \int e^{-z} \partial S,$$

$$q = \int r \, \partial x, \quad p = \int \partial x \int r \, \partial x,$$

et

$$y = \int p \, \partial x = \int \partial x \int \partial x \int r \, \partial x,$$

ubi omnia per solam variabilem s determinantur.

15. Quin etiam istas aequationes differentiales, quarum integralia hic exhibuimus, certo modo inter se coniungere licet, ut integratio eadem methodo, qua hic usi sumus, institui queat. Hoc modo nanciscemur innumera nova genera huiusmodi aequationum differentialium, quae etiam differentiando ad integrationem perduci poterunt, quod argumentum in sequentibus problematibus pertractemus.

PROBLEMA 9

16. *Posito $p + fq = t$, sint T et $\mathfrak{T}$ functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit haec aequatio differentialis secundi gradus: $y + fp + \mathfrak{T}x = T$, eius integrale completum investigare.*

SOLUTIO

Ponatur $y + fp = z$, erit

$$\partial z = \partial x(p + fq), \quad \text{ergo} \quad \partial z = t\partial x.$$

Quare cum nunc aequatio proposita sit $z + \mathfrak{T}x = T$, differentiando prodit

$$\partial z + \mathfrak{T}\partial x + x\partial\mathfrak{T} = \partial T,$$

sive

$$(t + \mathfrak{T})\partial x + x\partial\mathfrak{T} = \partial T,$$

unde colligitur haec aequatio:

$$\partial x + \frac{x\partial\mathfrak{T}}{t + \mathfrak{T}} = \frac{\partial T}{t + \mathfrak{T}},$$

ad quam integrandam ponatur

$$\int\frac{\partial t}{t + \mathfrak{T}} = u,$$

eritque

$$\int\frac{\partial\mathfrak{T}}{t + \mathfrak{T}} = l(t + \mathfrak{T}) - u,$$

tum vero aequatio nostra integrabilis reddetur, si eam multiplicemus per $e^{-u}(t + \mathfrak{T})$; integrale enim erit

$$xe^{-u}(t + \mathfrak{T}) = \int e^{-u}\partial T,$$

ex quo deducitur

$$x = \frac{e^u}{t + \mathfrak{T}}\int e^{-u}\partial T,$$

sicque x aequetur certae functioni ipsius t, quam hoc modo per integrationem invenire licet, eiusque differentiale erit

$$\partial x = \frac{\partial T - x\partial\mathfrak{T}}{t + \mathfrak{T}}.$$

Hinc igitur prodit $z = T - \mathfrak{T}x$. Cum nunc sit

$$y + fp = z, \quad \text{erit} \quad y\partial x + f\partial y = z\partial x,$$

unde colligitur

$$\partial y + \frac{y\partial x}{f} = \frac{z\partial x}{f},$$

quae aequatio multiplicata per $e^{\frac{x}{f}}$ dat integrale

$$ye^{\frac{x}{f}} = \frac{1}{f}\int e^{\frac{x}{f}} z\partial x,$$

ubi cum tam z quam x sint functiones ipsius t, erit etiam y functio ipsius t tantum, cum sit

$$y = \frac{e^{-\frac{x}{f}}}{f}\int e^{\frac{x}{f}} z\partial x.$$

PROBLEMA 10

17. *Posito $p + fq + gr = t$, si fuerint T et $\mathfrak{T}$ functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit haec aequatio differentialis tertii gradus:*

$$y + fp + gq + \mathfrak{T}x = T,$$

eius integrale completum invenire.

SOLUTIO

Ponatur

$$y + fp + gq = z,$$

eritque differentiando

$$\partial z = \partial x(p + fq + gr) = t\partial x,$$

sicque nostra aequatio integranda erit $z + \mathfrak{T}x = T$, pro qua erit ut ante

$$x = \frac{e^u}{t + \mathfrak{T}}\int e^{-u}\partial T, \quad \text{et} \quad z = T - \mathfrak{T}x,$$

posito scilicet $\int\frac{\partial t}{t + \mathfrak{T}} = u$. Ambae igitur illae expressiones functiones erunt solius variabilis t, unde etiam ∂x per eandem variabilem exprimetur. Tantum

igitur superest, ut etiam altera variabilis principalis y indagetur. Cum autem sit $y + fp + gq = z$, loco litterarum p et q scribantur valores initio assumti $\frac{\partial y}{\partial x}$ et $\frac{\partial\partial y}{\partial x^2}$, eritque, si tota aequatio per ∂x^2 multiplicetur, haec aequatio integranda:

$$y\,\partial x^2 + f\,\partial x\,\partial y + g\,\partial\partial y = z\,\partial x^2,$$

in qua cum tam x quam z sint functiones solius t, etiam y tanquam functionem ipsius t tractare licebit. Iam olim autem a me aliisque ostensum est, quomodo talis aequatio tractari debeat, quam ergo evolutionem hic repetere superfluum foret. Sufficiat enim notasse, valorem ipsius y per terminos huius formae $\int e^{\lambda x} z\,\partial x$ assignari, eum igitur per solam variabilem t exprimere licebit, sicque etiam y per functionem ipsius t definietur.

PROBLEMA 11

18. *Posito* $p + fq + gr + hs = t$, *si fuerint* T et $\mathfrak{T}$ *functiones quaecunque ipsius* t, *sive algebraicae sive transcendentes, ac proposita fuerit talis aequatio differentialis quarti gradus:*

$$y + fp + gq + hr + \mathfrak{T}x = T,$$

in eius integrale completum inquirere.

SOLUTIO

Sit $y + fp + gq + hr = z$, eritque differentiando

$$\partial z = \partial x(p + fq + gr + hs) = t\,\partial x,$$

atque aequatio integranda fiet $z + \mathfrak{T}x = T$, pro qua iterum, sumto

$$\int \frac{\partial t}{t + \mathfrak{T}} = u,$$

erit

$$x = \frac{e^u}{t + \mathfrak{T}} \int e^{-u}\partial T, \quad \text{atque} \quad z = T - \mathfrak{T}x,$$

ita ut tam x quam z per solam variabilem t exprimantur. His inventis, si in aequatione initio assumta loco p, q, r, s eorum valores substituantur, prodibit haec aequatio tertii gradus:

$$y\partial x^3 + f\partial x^2 \partial y + g\partial x \partial\partial y + h\partial^3 y = z\partial x^3,$$

cuius integrale completum per ea, quae circa huiusmodi aequationes sunt prolata, tanquam cognitum spectare licet, ita ut etiam hoc casu ambae variabiles x et y per novam variabilem t exprimantur. Facile autem patet hoc modo ad aequationes differentiales adhuc altiorum graduum progredi licere. Hac igitur ratione calculo integrali haud contemnendum incrementum allatum est censendum. Cum igitur hic praecipuum negotium versetur in integratione completa huiusmodi aequationis:

$$y + \frac{f\partial y}{\partial x} + \frac{g\partial\partial y}{\partial x^2} + \frac{h\partial^3 y}{\partial x^3} + \text{etc.} = z,$$

ubi z est functio quaecunque ipsius x, eius resolutionem iam passim exhibitam huc accommodemus et breviter ostendamus. Formetur haec aequatio:

$$1 + fu + gu^2 + hu^3 + iu^4 + \text{etc.} = 0,$$

cuius radices u designentur litteris α, β, γ, δ etc., quibus inventis erit uti iam olim ostendi[1])

$$y = \frac{e^{\alpha x}\int e^{-\alpha x} z\partial x}{f + 2g\alpha + 3h\alpha^2 + 4i\alpha^3 + \text{etc.}} + \frac{e^{\beta x}\int e^{-\beta x} z\partial x}{f + 2g\beta + 3h\beta^2 + 4i\beta^3 + \text{etc.}} + \text{etc.}$$

Hae scilicet formulae ex singulis radicibus α, β, γ, δ etc. formatae et iunctim sumtae dabunt valorem ipsius y atque adeo integrale completum, quia singulae formulae integrales constantem arbitrariam involvunt.

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1) Confer L. Euleri Commentationem 188, § 12; *Institutiones calculi integralis,* vol. II, § 1160. Vide notam 2) p. 251 huius voluminis. H. D.

SPECIMEN AEQUATIONUM DIFFERENTIALIUM INDEFINITI GRADUS EARUMQUE INTEGRATIONIS

M. S. Academiae exhibuit die 13. Decembris 1781
Commentatio 681 indicis ENESTROEMIANI
Institutiones calculi integralis 4, 1794, p. 577—589

1.[1]) Quando aequationes differentiales secundum gradus differentialium distinguuntur, ipsa rei natura gradus intermedios excludere videtur; cum enim totidem integrationibus opus sit, harum numerus certe non integer esse non potest. Incidi tamen nuper in aequationem differentialem indefiniti gradus, cuius exponens etiam numerus fractus esse potest, atque adeo mihi licuit eius integrale assignare; quod cum omni attentione dignum videatur, totam analysin, qua sum usus, hic dilucide exponam.

2. Cum miras proprietates unciarum potestatum binomii, quas hoc charactere indicare soleo $\left(\frac{p}{q}\right)$, cuius valor est hoc productum:

$$\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-2}{3}\cdots\frac{p-q+1}{q},$$

considerassem, in mentem mihi venit valorem huiusmodi formulae $\left(\frac{p}{q}\right)$ ad formulam integralem revocare, unde etiam casus, quibus p et q non sunt numeri integri, assignari queant. Directe quidem talem reductionem non succedere observavi, unde eius valorem reciprocum $\frac{1}{\left(\frac{p}{q}\right)}$ sum contemplatus, cuius valor est

1) In editione principe paragraphi numeris 19 usque ad 40 signatae sunt. H. D.

$$\frac{1}{p}\cdot\frac{2}{p-1}\cdot\frac{3}{p-2}\cdots\frac{q}{p-q+1}.$$

Hunc in finem statuo

$$\frac{1\cdot 2\cdot 3\cdots q\times x^p}{p(p-1)(p-2)\cdots(p-q+1)}=s,$$

ita ut posito $x=1$ desideratus valor ipsius $1:\left(\frac{p}{q}\right)$ obtineatur.

3. Sit nunc brevitatis gratia $1\cdot 2\cdot 3\cdots q=N$, ut habeatur

$$s=\frac{Nx^p}{p\cdots(p-q+1)},$$

in cuius denominatore tenendum est factores continuo unitate decrescere. Quod si iam ista formula differentietur, prodibit

$$\frac{\partial s}{\partial x}=\frac{Nx^{p-1}}{(p-1)\cdots(p-q+1)},$$

sicque primus factor denominatoris est sublatus, ac differentiatione denuo instituta prodibit

$$\frac{\partial\partial s}{\partial x^2}=\frac{Nx^{p-2}}{(p-2)\cdots(p-q+1)}.$$

Hoc igitur modo continuo differentiando omnes factores denominatoris tollentur, ac pervenietur tandem ad hanc aequationem:

$$\frac{\partial^q s}{\partial x^q}=Nx^{p-q}.$$

4. Pervenimus igitur, loco N valorem suum substituendo, ad hanc aequationem differentialem

$$\frac{\partial^q s}{1\cdots q\,\partial x^q}=x^{p-q},$$

quam ergo tot vicibus integrari oporteret, quot q continet unitates, atque singulae integrationes ita sunt instituendae, ut posito $x=0$ integralia evanescant, et postquam omnes integrationes fuerint absolutae, loco x scribi debebit unitas, hocque modo valor ipsius s resultans dabit valorem formulae $1:\left(\frac{p}{q}\right)$.

Quo autem istas integrationes generalius expediamus, loco x^{p-q} scribamus X, ut habeamus hanc aequationem resolvendam

$$\frac{\partial^q s}{1\cdot 2\cdots q\partial x^q}=X.$$

5. Hanc aequationem primo multiplicemus per ∂x, eiusque integrale dabit

$$\frac{\partial^{q-1} s}{1\cdot 2\cdot 3\cdots q\partial x^{q-1}}=\int X\partial x.$$

Istam aequationem ducamus in $1\cdot\partial x$, eritque integrando

$$\frac{\partial^{q-2} s}{2\cdot 3\cdots q\cdot\partial x^{q-2}}=\int\partial x\int X\partial x=x\int X\partial x-\int Xx\partial x.$$

Per notas enim reductiones eiusmodi integralia repetita ad simplicia reduci possunt[1]). Haec aequatio iam per $2\partial x$ multiplicata eodemque modo integrata praebebit

$$\frac{\partial^{q-3} s}{3\cdot 4\cdots q\cdot\partial x^{q-3}}=x^2\int X\partial x-2x\int Xx\partial x+\int Xx^2\partial x.$$

Nunc per $3\partial x$ multiplicando et integrando proveniet

$$\frac{\partial^{q-4} s}{4\cdot 5\cdots q\cdot\partial x^{q-4}}=x^3\int X\partial x-3x^2\int Xx\partial x+3x\int Xx^2\partial x-\int Xx^3\partial x.$$

Eodem modo reperietur

$$\frac{\partial^{q-5} s}{5\cdot 6\cdots q\cdot\partial x^{q-5}}=x^4\int X\partial x-4x^3\int Xx\partial x+6x^2\int Xx^2\partial x-4x\int Xx^3\partial x+\int Xx^4\partial x,$$

sicque in genere nostros characteres in usum vocando erit

$$\frac{\partial^{q-n} s}{n(n+1)\cdots q\cdot\partial x^{q-n}}=$$

$$x^{n-1}\int X\partial x-\left(\frac{n-1}{1}\right)x^{n-2}\int Xx\partial x+\left(\frac{n-1}{2}\right)x^{n-3}\int Xx^2\partial x-\left(\frac{n-1}{3}\right)x^{n-4}\int Xx^3\partial x+\text{ etc.}$$

6. Statuamus nunc $n=q$, et cum sit $\partial^0 s=s$, orietur haec aequatio finita:

$$\frac{s}{q}=x^{q-1}\int X\partial x-\left(\frac{q-1}{1}\right)x^{q-2}\int Xx\partial x+\left(\frac{q-1}{2}\right)x^{q-3}\int Xx^2\partial x-\text{ etc.},$$

1) Confer Commentationem 679 huius voluminis. H. D.

cuius singula membra ita integrari debent, ut posito $x = 0$ evanescant, quod quidem semper eveniet, si modo sit $q - 1 > 0$, quamobrem ipsae formulae integrales $\int X \partial x$, $\int X x \partial x$ etc. tantum sine adiectione constantis integrari debent. Etsi enim hoc modo x forte in denominatorem ingrediatur, per potestatem ipsius x, qua multiplicari debent, iterum tolletur.

7. His circa singula integralia observatis extra signa summatoria iam ponere licebit $x = 1$, quippe qui est casus quaestionis propositae; sicque reperietur

$$1 : q\left(\frac{p}{q}\right) = \int X \partial x \left[1 - \left(\frac{q-1}{1}\right)x + \left(\frac{q-1}{2}\right)x^2 - \left(\frac{q-1}{3}\right)x^3 + \text{etc.}\right],$$

cuius seriei valor manifesto est $(1-x)^{q-1}$, ita ut habeamus hanc expressionem determinatam:

$$\frac{1}{q\left(\frac{p}{q}\right)} = \int X \partial x (1-x)^{q-1},$$

cuius ergo valor etiam casibus, quibus q non est numerus integer, per quadraturas exhiberi potest, sicque aequationis differentialis indefiniti gradus $\partial^q s = N X \partial x^q$ integrale feliciter elicuimus, et quia $X = x^{p-q}$, omnes unciae hoc modo ad formas integrales redigentur

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{p-q} \partial x \, (1-x)^{q-1}},$$

et quia exponentes ipsius x et ipsius $1 - x$ permutari possunt, erit etiam

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{q-1} \partial x \, (1-x)^{p-q}},$$

hancque formulam ex principio diversissimo non ita pridem sum adeptus[1]).

THEOREMA

8. Valor huius characteris $\left(\frac{p}{q}\right)$ reduci potest ad formulam integralem, cum sit

1) Vide L. Euleri Commentationem 421 (indicis *Enestroemiani*). *Evolutio formulae integralis* $\int x^{f-1} dx (lx)^{\frac{m}{n}}$, *integratione a valore* $x = 0$ *ad* $x = 1$ *extensa.* Novi Comment. acad. sc. Petrop. **16**, 1772, p. 92. *Leonhardi Euleri Opera omnia*, series I, vol. 17. H. D.

$$\left(\frac{p}{q}\right) = \frac{1}{q\int x^{q-1}\partial x\,(1-x)^{p-q}},$$

siquidem hoc integrale ab $x = 0$ ad $x = 1$ extendatur.

COROLLARIUM 1

9. Sumto ergo $p = 0$ erit

$$\left(\frac{0}{q}\right) = \frac{1}{q\int x^{q-1}\partial x\,(1-x)^{-q}}.$$

Ostendi autem olim[1]) esse

$$\int x^{q-1}\partial x(1-x)^{-q} = \frac{\pi}{\sin.\pi q},$$

unde ergo fiet

$$\left(\frac{0}{q}\right) = \frac{\sin.\pi q}{\pi q}.$$

COROLLARIUM 2

10. Deinde per notam[2]) integralium reductionem reperitur

$$\int x^{q-1}\partial x(1-x)^{p-q} = \frac{\pi}{\sin.\pi q}\left(\frac{p-q}{q}\right),$$

cuius ergo valor, quoties p est numerus integer, absolute assignari potest, quamobrem in genere erit

$$\left(\frac{p}{q}\right) = \frac{\sin.\pi q}{\pi q} : \left(\frac{p-q}{q}\right).$$

---

1) L. EULERI Commentatio 321 (indicis *Enestroemiani*): *Observationes circa integralia formularum* $\int x^{p-1}dx(1-x^n)^{\frac{q}{n}-1}$ *posito post integrationem* $x = 1$. Mél. Turin 3, 1766, p. 156. Vide quoque *Institutionum calculi integralis* vol. I, § 351 et 368 et Commentationes 499, 588, 640: *De integratione formulae* $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$, *ab* $x=0$ *ad* $x=1$ *extensa*. Acta acad. sc. Petrop. 1777, II, 1780, p. 3. *Investigatio formulae integralis* $\int \frac{x^{m-1}dx}{(1+x^k)^n}$, *casu quo post integrationem statuitur* $x = \infty$. Opuscula analytica 2, 1785, p. 42. *Comparatio valorum formulae integralis* $\int \frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}$ *a termino* $x = 0$ *usque ad* $x = 1$ *extensae*. Nova acta acad. sc. Petrop. 5, 1789, p. 86. LEONHARDI EULERI *Opera omnia*, series I, vol. 11, 17, 18. H. D.

2) Vide *Institutionum calculi integralis* vol. I, § 118, 345. LEONHARDI EULERI *Opera omnia*, series I, vol. 11. H. D.

COROLLARIUM 3

11. Cum igitur vicissim sit

$$\int x^{q-1}\partial x(1-x)^{p-q}=\frac{1}{q\left(\frac{p}{q}\right)},$$

si hic loco $q-1$ scribamus f, et g loco $p-q$, habebimus

$$\int x^f\partial x(1-x)^g=\frac{1}{(1+f)\left(\frac{f+g+1}{f+1}\right)}.$$

SCHOLION

12. Quoniam igitur hanc formulam integralem nacti sumus ex aequatione integrali indefiniti gradus, eandem investigationem latius extendamus in sequente problemate.

PROBLEMA

13. *Proposita serie sive finita sive infinita*

$$S=\frac{A}{\left(\frac{p}{q}\right)}+\frac{B}{\left(\frac{p+1}{q}\right)}+\frac{C}{\left(\frac{p+2}{q}\right)}+\frac{D}{\left(\frac{p+3}{q}\right)}+\textit{etc.}$$

eius valorem per formulam integralem exprimere[1]).

SOLUTIO

Tribuamus singulis terminis potestates ipsius x, ac statuamus

$$S=\frac{Ax^p}{\left(\frac{p}{q}\right)}+\frac{Bx^{p+1}}{\left(\frac{p+1}{q}\right)}+\frac{Cx^{p+2}}{\left(\frac{p+2}{q}\right)}+\text{etc.},$$

1) Ea quae sequuntur non satis recte demonstrantur, sed vera sunt, exceptis iis in quibus Eulerus seriebus divergentibus utitur. Oportet uncias ut analyticas functiones accipi, quae pro certis valoribus ipsorum p et q integrali informantur, sed pro ceteris valoribus eundem valorem praebent, qui functionem per integrale definitam analytice prolongando efficitur. H. D.

quae series ergo, posito $x = 1$, praebebit ipsam seriem propositam. Ubi observandum, in omnibus terminis litteram q eundem retinere valorem, alteram vero p continuo unitate augeri, unde productum indefinitum $1 \cdot 2 \cdot 3 \cdots q = N$ in omnibus terminis eundem retinebit valorem. Quare cum supra ex aequatione $s = \frac{x^p}{\left(\frac{p}{q}\right)}$ deduxerimus hanc aequationem differentialem indefiniti gradus:

$$\frac{\partial^q s}{\partial x^q} = N x^{p-q},$$

ex singulis terminis nostrae seriei idem resultabit differentiale, si modo exponentem p unitate augeamus, unde ergo reperiemus

$$\frac{\partial^q S}{\partial x^q} = NAx^{p-q} + NBx^{p-q+1} + \text{etc.}$$

14. Ponamus nunc

$$A + Bx + Cx^2 + Dx^3 + \text{etc.} = V,$$

eritque

$$\frac{\partial^q S}{N\,\partial x^q} = x^{p-q}V,$$

quamobrem si statuamus $x^{p-q}V = X$, habebimus ipsam aequationem iam ante tractatam

$$\frac{\partial^q S}{1 \cdot 2 \cdots q\,\partial x^q} = X,$$

cuius integratio q vicibus repetita nos perduxit ad hanc expressionem $S = q\int X\partial x(1 - x)^{q-1}$, unde ergo pro X et V valores substituendo nanciscemur summam quaesitam S, scilicet

$$S = q\int x^{p-q}\partial x(A + Bx + Cx^2 + Dx^3 + \text{etc.})\,(1 - x)^{q-1},$$

si modo hoc integrale ab $x = 0$ ad $x = 1$ extendatur, vel ut ante invenimus, si modo in integratione nulla constans adiiciatur, deinde vero sumatur $x = 1$.

EXEMPLUM

15. Sit $V = (1-x)^n$, ita ut sit

$$A=1,\ B=-\left(\frac{n}{1}\right),\quad C=+\left(\frac{n}{2}\right),\quad D=-\left(\frac{n}{3}\right)\ \text{etc.},$$

et series proposita erit

$$S=\frac{1}{\left(\frac{p}{q}\right)}-\frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)}-\frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)}+\text{etc.},$$

tum igitur summa huius seriei erit

$$S=q\int x^{p-q}\partial x(1-x)^{q+n-1},$$

sive permutatis exponentibus ipsius x et $1-x$, erit quoque

$$S=q\int x^{q+n-1}\partial x(1-x)^{p-q}.$$

Nunc autem evidens est hanc ipsam formulam integralem iterum ad characterem hic usitatum reduci posse; ope § 11 erit enim $f=q+n-1$ et $g=p-q$, atque hinc prodibit

$$S=\frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Hinc ergo sive formulis integralibus habebimus hanc summationem seriei infinitae maxime notabilem:

$$\frac{1}{\left(\frac{p}{q}\right)}-\frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)}-\frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)}+\frac{\left(\frac{n}{4}\right)}{\left(\frac{p+4}{q}\right)}-\text{etc.}=\frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

COROLLARIUM 1

16. Si ergo fuerit $n=0$, oritur aequatio manifeste identica scilicet $\frac{1}{\left(\frac{p}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}$. At si $n=1$ prodit

$$\frac{q}{(q+1)\binom{p+1}{q+1}}=\frac{1}{\binom{p}{q}}-\frac{1}{\binom{p+1}{q}}.$$ [1)]

Si $n=2$, fiet

$$\frac{q}{(q+2)\binom{p+2}{q+2}}=\frac{1}{\binom{p}{q}}-\frac{2}{\binom{p+1}{q}}+\frac{1}{\binom{p+2}{q}}.$$

COROLLARIUM 2

17. Quo consensus cum veritate clarius appareat, evolvamus casum determinatum, quo $p=3$, $q=2$, $n=4$, eritque

$$\frac{q}{q+n}=\frac{1}{3},\quad \text{et}\quad \binom{p+n}{q+n}=\binom{7}{6}=\binom{7}{1}=7.$$

Deinde fit

$$\binom{p}{q}=\binom{3}{2}=3,\quad \binom{p+1}{q}=\binom{4}{2}=6,\quad \binom{p+2}{q}=\binom{5}{2}=10,\quad \binom{p+3}{q}=\binom{6}{2}=15,$$

quae est progressio numerorum trigonalium; tum vero erit

$$\binom{n}{1}=4,\quad \binom{n}{2}=6,\quad \binom{n}{3}=4,\quad \binom{n}{4}=1.$$

His igitur valoribus substitutis erit

$$\frac{1}{3\cdot 7}=\frac{1}{3}-\frac{4}{6}+\frac{6}{10}-\frac{4}{15}+\frac{1}{21},$$

quod egregie convenit.

EXEMPLUM 2

18. Statuamus $V=(1+x)^{q-1}$, ut fiat

$$S=q\int x^{p-q}\,\partial x\,(1-xx)^{q-1};$$

tum vero erit

1) Editio princeps loco $\binom{p+1}{q}$ habet $\binom{p+1}{q+1}$. Correxit H. D.

$$A=1,\quad B=\left(\frac{q-1}{1}\right),\quad C=\left(\frac{q-1}{2}\right),\quad D=\left(\frac{q-1}{3}\right) \text{ etc.},$$

sicque series proposita erit

$$S=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\frac{\left(\frac{q-1}{3}\right)}{\left(\frac{p+3}{q}\right)}+\text{ etc.}$$

Evidens autem est, hanc formulam integralem etiam ad nostros characteres reduci posse. Ponamus enim $xx=y$, erit

$$S=\frac{q}{2}\int y^{\frac{p-q-1}{2}}\partial y(1-y)^{q-1},$$

sive permutatis exponentibus

$$S=\frac{q}{2}\int y^{q-1}\partial y(1-y)^{\frac{p-q-1}{2}},$$

quae comparata cum § 11 dat $f=q-1$, $g=\frac{p-q-1}{2}$, quibus valoribus substitutis colligitur

$$S=\frac{q}{2q\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{2\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{ etc.},$$

vel si ponatur $\frac{p+q-1}{2}=r$, erit

$$S=\frac{1}{2\left(\frac{r}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{ etc.}$$

COROLLARIUM 1

19. Hic casu $q=1$ summa inventa ipsi termino primo aequatur. Sumamus autem $q=2$, erit

$$\frac{1}{2\left(\frac{\frac{p+1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{2}\right)}+\frac{1}{\left(\frac{p+1}{2}\right)},$$

hoc est

$$\frac{4}{pp-1}=\frac{2}{p(p-1)}+\frac{2}{p(p+1)},$$

unde patet istam summationem esse veritati consentaneam, de quo quidem nullum superesse potest dubium, quoties q est numerus integer positivus; quamobrem quosdam casus consideremus, ubi non est talis.

COROLLARIUM 2

20. Quo autem evolutio facilior evadat, contemplemur casum quo $r=q$, ut fiat $\left(\frac{r}{q}\right)=1$, tum autem erit $p=1+q$ hincque

$$\left(\frac{p}{q}\right)=1+q,\quad \left(\frac{p+1}{q}\right)=\frac{q+1}{1}\cdot\frac{q+2}{2},\quad \left(\frac{p+2}{q}\right)=\frac{q+1}{1}\cdot\frac{q+2}{2}\cdot\frac{q+3}{3},$$

quibus substitutis orietur haec series:

$$\frac{1}{2}=\frac{1}{q+1}+\frac{2(q-1)}{(q+1)(q+2)}+\frac{3(q-1)(q-2)}{(q+1)(q+2)(q+3)}+\frac{4(q-1)(q-2)(q-3)}{(q+1)(q+2)(q+3)(q+4)}+\text{etc.}^{1)},$$

quae series notatu maxime est digna, quia eius summa semper est $\frac{1}{2}$, quicunque valores litterae q tribuantur. Si enim sit $q=0$, habebitur

$$\tfrac{1}{2}=1-1+1-1+1-\text{etc.},$$

quae est series notissima[2]). Sit nunc $q=-1$, et ob $q+1=0$ multiplicemus omnes terminos per $q+1$, prodibitque haec series

$$0=1-4+9-16+25-\text{etc.},$$

uti differentias sumendo[2]) facile patet. Ponamus $q=\frac{1}{2}$, et haec series prodibit:

$$\frac{1}{2}=\frac{2}{3}-\frac{2\cdot 2}{3\cdot 5}+\frac{2\cdot 3}{5\cdot 7}-\frac{2\cdot 4}{7\cdot 9}+\frac{2\cdot 5}{9\cdot 11}-\text{etc.}$$

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1) Haec series non convergit si $q\leq 0$. H. D.

2) Vide *Institutionum calculi differentialis* partem posteriorem, § 9. *LEONHARDI EULERI Opera omnia*, series I, vol. 10. H. D.

Cum igitur sit

$$\frac{2}{3}=1-\frac{1}{3},\quad \frac{4}{3\cdot 5}=\frac{2}{3}-\frac{2}{5},\quad \frac{6}{5\cdot 7}=\frac{3}{5}-\frac{3}{7},\quad \frac{8}{7\cdot 9}=\frac{4}{7}-\frac{4}{9},$$

et ita porro, his substitutis prodibit haec series:

$$\tfrac{1}{2}=1-1+1-1+1-1+1-\text{etc.}$$

At si sumamus $q=-\frac{1}{2}$ erit

$$\tfrac{1}{2}=2-4+6-8+10-12+\text{etc.},$$

quod per differentias fit manifestum.

COROLLARIUM 3

21. Sumamus nunc $r=0$, ut fiat $p=1-q$. Demonstravi autem esse $\left(\frac{0}{q}\right)=\frac{\sin. q\pi}{q\pi}$, unde orietur

$$\frac{\pi q}{2\sin.\pi q}=\frac{1}{\left(\frac{1-q}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{2-q}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{3-q}{q}\right)}+\text{etc.}^{1)},$$

cuius casum $q=\frac{1}{2}$ evolvisse pretium erit, membrum enim sinistrum fit $\frac{\pi}{4}$. Pro parte dextra autem habebimus

$$\left(\frac{q-1}{1}\right)=-\frac{1}{2},\quad \left(\frac{q-1}{2}\right)=\frac{1\cdot 3}{2\cdot 4},\quad \left(\frac{q-1}{3}\right)=-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\ \text{etc.},$$

tum vero pro denominatoribus

$$\left(\frac{1-q}{q}\right)=1,\quad \left(\frac{2-q}{q}\right)=\frac{3}{2},\quad \left(\frac{3-q}{q}\right)=\frac{3\cdot 5}{2\cdot 4},\quad \left(\frac{4-q}{q}\right)=\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\ \text{etc.},$$

quibus valoribus substitutis orietur haec series:

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.},$$

quae est series notissima. Ponamus autem adhuc $q=-\frac{1}{2}$, et membrum sinistrum erit ut ante $\frac{\pi}{4}$; pro parte dextra autem erit

1) Haec series non convergit, si $q<0$. H. D.

$$\left(\frac{q-1}{1}\right)=-\frac{3}{2},\quad \left(\frac{q-1}{2}\right)=\frac{3\cdot 5}{2\cdot 4},\quad \left(\frac{q-1}{3}\right)=-\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\ \text{etc.}$$

tum

$$\left(\frac{1-q}{q}\right)=\frac{1\cdot 3}{2\cdot 4},\quad \left(\frac{2-q}{q}\right)=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},\quad \left(\frac{3-q}{q}\right)=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\ \text{etc.}$$

hinc

$$\frac{\pi}{4}=\frac{2\cdot 4}{1\cdot 3}-\frac{4\cdot 6}{1\cdot 5}+\frac{6\cdot 8}{1\cdot 7}-\frac{8\cdot 10}{1\cdot 9}+\text{etc.},$$

cuius veritas ita ostenditur. Cum sit

$$\frac{2\cdot 4}{1\cdot 3}=3-\frac{1}{3},\quad \frac{4\cdot 6}{1\cdot 5}=5-\frac{1}{5},\quad \frac{6\cdot 8}{1\cdot 7}=7-\frac{1}{7},\quad \frac{8\cdot 10}{1\cdot 9}=9-\frac{1}{9}\ \text{etc.},$$

erit illa series aequalis huic:

$$\frac{\pi}{4}=3-\frac{1}{3}-5+\frac{1}{5}+7-\frac{1}{7}-9+\frac{1}{9}+\text{etc.},$$

quae series in has duas discerpatur:

$$\frac{\pi}{4}=\begin{cases}\quad 3-5+7-9+11-13+\text{etc.}\\ -\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}\end{cases}$$

De superiore notetur, eius summam per differentias erutam esse

$$3-5+7-9+11-13+\text{etc.}=1;$$

inferioris summa ex serie supra inventa, qua erat

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.},$$

erit

$$-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.}=\frac{\pi}{4}-1,$$

unde iam manifestum est fore

$$3-\frac{1}{3}-5+\frac{1}{5}+7-\frac{1}{7}-9+\frac{1}{9}+\text{ etc.}=1+\frac{\pi}{4}-1=\frac{\pi}{4}.$$

Hinc igitur patet, pro q etiam numeros negativos atque adeo fractos accipi posse.

THEOREMA GENERALE

22. Si X denotet functionem quamcunque ipsius x, et proposita fuerit haec aequatio differentialis cuiuscunque gradus:

$$\partial^q y = 1\cdot 2\cdot 3\cdots qX\partial x^q,$$

ubi exponens q denotet numeros quoscunque sive integros sive fractos sive positivos sive negativos, cuius ergo aequationis resolutio totidem integrationes requirit, quae si singulae ab $x=0$ inchoentur omnibusque peractis statuatur $x=1$, tum semper erit $y=q\int X\partial x(1-x)^{q-1}$, hoc scilicet integrali ab $x=0$ ad $x=1$ extenso[1]).

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1) Cf. notam formulam $y=q\int_0^x(x-z)^{q-1}x(z)\,dz$, quae valorem ipsius y non solum pro $x=1$, sed etiam pro omni valore praebet. Haec formula reipsa demonstrata est § 7. H. D.

DE INSIGNIBUS PROPRIETATIBUS FORMULARUM INTEGRALIUM PRAETER BINAS VARIABILES ETIAM EARUM DIFFERENTIALIA CUIUSCUNQUE ORDINIS INVOLVENTIUM

Conventui exhibita die 10. Martii 1777

Commentatio 687 indicis Enestroemiani

Nova acta academiae scientiarum Petropolitanae 9 (1791), 1795, p. 81—97

Summarium ibidem p. 170—171

SUMMARIUM

Soit Z une fonction à deux variables x et y seulement, mais qui renferme, outre ces deux variables aussi leurs différentielles d'un ordre quelconque, si l'on dégage cette fonction des différentielles, en mettant

$$\partial y = p\,\partial x, \;\; \partial p = q\,\partial x, \;\; \partial q = r\,\partial x, \;\; \partial r = s\,\partial x \text{ etc.}$$

sa différentielle sera $\partial Z = V\partial x$, V étant une fonction de x, y, p, q, r etc. dépendante de Z, savoir

$$V = \left(\frac{\partial Z}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial Z}{\partial p}\right) + r\left(\frac{\partial Z}{\partial q}\right) + \text{etc.}$$

De là il suit que si V marque une fonction telle qu'on vient de dire, la formule différentielle $V\partial x$ sera toujours intégrable, quoique x et y fussent entièrement indépendantes l'une de l'autre, au lieu que si pour V on prenoit une autre fonction des variables x, y, p, q, r etc., l'intégration ne sauroit avoir lieu, à moins qu'il n'y eût une certaine relation entre les variables x et y.

Cela remarqué M. Euler fait voir dans une suite de Théorèmes que si la formule $V\partial x$ est intégrable, toutes les formules suivantes

$$\partial x\left(\frac{\partial V}{\partial x}\right),\ \partial x\left(\frac{\partial V}{\partial y}\right),\ \partial x\left(\frac{\partial V}{\partial p}\right),\ \partial x\left(\frac{\partial V}{\partial q}\right),\ \partial x\left(\frac{\partial V}{\partial r}\right),\ \partial x\left(\frac{\partial\partial V}{\partial x^2}\right),\ \partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right),\ \partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right),\ \partial x\left(\frac{\partial\partial V}{\partial x\partial q}\right),$$
$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial p}\right),\ \partial x\left(\frac{\partial\partial V}{\partial y\,\partial q}\right)\ \text{etc.}$$

le seront aussi, et qu'en général toutes les fois que $V\partial x$ admet l'intégration, la formule

$$\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\text{ etc.}}\ V}{\partial x^\alpha\,\partial y^\beta\,\partial p^\gamma\,\partial q^\delta\ \text{etc.}}\right)$$

sera pareillement intégrable[1]).

Au reste on trouve dans ce mémoire une nouvelle démonstration du critère général et connu de l'intégrabilité de la formule $V\partial x$, qui ne diffère pas beaucoup de celle, que feu M. LEXELL en avoit donnée dans les nouveaux commentaires de l'Académie.

1. Si Z fuerit functio quaecunque, non solum binas variabiles x et y, sed etiam earum differentialia cuiuscunque ordinis involvens, ea saltem a specie differentialium liberari potest ope sequentium positionum:

$$\partial y = p\,\partial x,\quad \partial p = q\,\partial x,\quad \partial q = r\,\partial x,\quad \partial r = s\,\partial x,\quad \partial s = t\,\partial x\quad \text{etc.};$$

tum enim his valoribus substitutis quantitas Z, si fuerit finita, evadet functio quantitatum finitarum x, y, p, q, r, s, t etc. Ita si fuerit

$$Z = \frac{(\partial x^2 + \partial y^2)^{\frac{3}{2}}}{\partial x\,\partial\partial y - \partial y\,\partial\partial x},$$

quae est formula notissima pro radio osculi, ob

$$\partial y = p\,\partial x \quad \text{et} \quad \partial\partial y = p\,\partial\partial x + \partial p\,\partial x = p\,\partial\partial x + q\,\partial x^2,$$

primo numerator hanc induet formam: $\partial x^3(1+pp)^{\frac{3}{2}}$, deinde vero denominator evadet $= q\,\partial x^3$, sicque ista quantitas erit

$$Z = \frac{(1+pp)^{\frac{3}{2}}}{q}.$$

1) Enoncé en partie inexact. D'après le théorème énoncé § 29 ce sont les expressions telles que

$$\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}\,V}{\partial x^\alpha\,\partial y^\beta\,\partial p^\gamma\,\partial q^\delta} - \gamma\frac{\partial^{\alpha+\beta+\gamma+\delta}\,Z}{\partial x^\alpha\,\partial y^{\beta+1}\,\partial p^{\gamma-1}\,\partial q^\delta} - \delta\frac{\partial^{\alpha+\beta+\gamma+\delta}\,Z}{\partial x^\alpha\,\partial y^\beta\,\partial p^{\gamma+1}\,\partial q^{\delta-1}}\right)$$

qui sont intégrables. H. D.

2. Quodsi nunc talis functio Z differentietur, eius differentiale ex tot constabit partibus, quot in ea insunt litterarum x, y, p, q, r, s etc., ideoque tali forma exprimetur[1]):

$$\partial Z = M\partial x + N\partial y + P\partial p + Q\,\partial q + R\,\partial r + S\,\partial s + \text{etc.}$$

Hic autem, ne multitudo litterarum M, N, P, Q, R etc. in calculo molestiam creet, eas, quoniam omnes pendent a natura functionis Z, per sequentes characteres usu iam satis receptos repraesentabo:

$$M = \left(\frac{\partial Z}{\partial x}\right),\quad N = \left(\frac{\partial Z}{\partial y}\right),\quad P = \left(\frac{\partial Z}{\partial p}\right),\quad Q = \left(\frac{\partial Z}{\partial q}\right),\quad R = \left(\frac{\partial Z}{\partial r}\right),\quad S = \left(\frac{\partial Z}{\partial s}\right)\ \text{etc.}$$

hocque modo, nullas litteras peregrinas introducendo, erit

$$\partial Z = \partial x\left(\frac{\partial Z}{\partial x}\right) + \partial y\left(\frac{\partial Z}{\partial y}\right) + \partial p\left(\frac{\partial Z}{\partial p}\right) + \partial q\left(\frac{\partial Z}{\partial q}\right) + \text{etc.},$$

ac si porro loco differentialium ∂y, ∂p, ∂q, ∂r etc. valores supra assignatos adhibeamus, prodibit

$$\partial Z = \partial x\left(\frac{\partial Z}{\partial x}\right) + p\partial x\left(\frac{\partial Z}{\partial y}\right) + q\partial x\left(\frac{\partial Z}{\partial p}\right) + r\partial x\left(\frac{\partial Z}{\partial q}\right) + \text{etc.}$$

3. Hinc ergo si statuamus

$$V = \left(\frac{\partial Z}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial Z}{\partial p}\right) + r\left(\frac{\partial Z}{\partial q}\right) + \text{etc.},$$

quae erit quantitas finita, pariterque certa functio ipsarum x, y, p, q, r etc. ab indole functionis Z pendens, erit

$$\partial Z = V\partial x,\quad \text{ideoque integrando}\quad Z = \int V\partial x,$$

in qua integratione omnes litterae x, y, p, q, r etc. tanquam variabiles insunt. Ubi probe notetur, si V fuerit talis functio, qualem descripsimus, tum formulam differentialem $V\partial x$ semper integrationem admittere, etiamsi binae variabiles

1) In formulis: ∂Z, $\partial \cdot \left(\frac{\partial Z}{\partial x}\right)$ etc. ∂ sine denominatore significat differentiale plenum. H. D.

x et y nullo modo a se invicem pendeant, cum contra, si loco V alia quaecunque functio quantitatum x, y, p, q, r etc. acciperetur, integratio locum habere non posset, nisi certa quaedam relatio inter binas variabiles x et y statueretur.

4. His constitutis cum sit

$$V = \left(\frac{\partial Z}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial Z}{\partial p}\right) + r\left(\frac{\partial Z}{\partial q}\right) + \text{etc.},$$

perpendamus valores differentiales ipsius V, qui oriuntur, si vel sola quantitas x, vel sola y, vel sola p, vel sola q etc. pro variabili habeatur, quos valores simili ratione per hos characteres:

$$\left(\frac{\partial V}{\partial x}\right), \quad \left(\frac{\partial V}{\partial y}\right), \quad \left(\frac{\partial V}{\partial p}\right), \quad \left(\frac{\partial V}{\partial q}\right) \quad \text{etc.}$$

designemus. Ac primo quidem si sola quantitas x ut variabilis tractetur, iisdem characteribus adhibendis reperietur:

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial\partial Z}{\partial x^2}\right) + p\left(\frac{\partial\partial Z}{\partial y\,\partial x}\right) + q\left(\frac{\partial\partial Z}{\partial p\,\partial x}\right) + r\left(\frac{\partial\partial Z}{\partial q\,\partial x}\right) + s\left(\frac{\partial\partial Z}{\partial r\,\partial x}\right) + \text{etc.},$$

ubi scilicet, uti iam satis est usu receptum, formula $\left(\frac{\partial\partial Z}{\partial x^2}\right)$ indicat, functionem Z bis differentiandam esse, sola x pro variabili assumta; at formula $\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right)$ indicat, functionem Z etiam bis ita esse differentiandam, ut in altera differentiatione sola quantitas x, in altera vero sola y variabilis sumatur. Demonstratum autem est eundem valorem prodire, sive in prima operatione x, in secunda vero y, sive inverso modo, in prima y in altera vero x variabilis statuatur; quod idem etiam de reliquis formulis duplicem differentiationem innuentibus est tenendum.

5. Si iam in hac postrema expressione valorem $\left(\frac{\partial Z}{\partial x}\right)$ littera T designemus, hinc fiet

$$\left(\frac{\partial\partial Z}{\partial x^2}\right) = \frac{\partial T}{\partial x};$$

tum vero

$$\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right) = \frac{\partial T}{\partial y}, \quad \left(\frac{\partial\partial Z}{\partial x\,\partial p}\right) = \frac{\partial T}{\partial p} \quad \text{etc.}$$

hisque formulis introductis erit

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.}$$

At vero si quantitas ista T per variabilitatem omnium litterarum x, y, p, q, r etc. differentietur, erit, ut supra iam vidimus, eius differentiale plenum:

$$\partial T = \partial x\left(\frac{\partial T}{\partial x}\right) + p\partial x\left(\frac{\partial T}{\partial y}\right) + q\partial x\left(\frac{\partial T}{\partial p}\right) + r\partial x\left(\frac{\partial T}{\partial q}\right) + \text{etc.},$$

unde patet fore

$$\partial T = \partial x\left(\frac{\partial V}{\partial x}\right),$$

ita ut integrando sit

$$T = \left(\frac{\partial Z}{\partial x}\right) = \int \partial x\left(\frac{\partial V}{\partial x}\right).$$

Hinc discimus, si formula $V\partial x$ integrationem admittat, semper etiam hanc formulam $\partial x\left(\frac{\partial V}{\partial x}\right)$ integrationem esse admissuram; quam proprietatem hoc Theoremate 1 referamus.

Si fuerit $\int V\partial x = Z$, *tum etiam semper erit*

$$\int \partial x\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial Z}{\partial x}\right), \quad \textit{sive} \quad \partial x\left(\frac{\partial V}{\partial x}\right) = \partial \cdot \left(\frac{\partial Z}{\partial x}\right).$$

6. Nunc quantitatis V id consideremus differentiale, quod ex sola variabili y enascitur, ac reperietur

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial\partial Z}{\partial x\,\partial y}\right) + p\left(\frac{\partial\partial Z}{\partial y^2}\right) + q\left(\frac{\partial\partial Z}{\partial p\,\partial y}\right) + r\left(\frac{\partial\partial Z}{\partial q\,\partial y}\right) + \text{etc.},$$

unde si hic ponamus $\left(\frac{\partial Z}{\partial y}\right) = T$, erit

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.},$$

hinc igitur, ut supra patet, fore

$$\partial x\left(\frac{\partial V}{\partial y}\right)=\partial T=\partial\cdot\left(\frac{\partial Z}{\partial y}\right),$$

ex quo integrando erit

$$\int\partial x\left(\frac{\partial V}{\partial y}\right)=T=\left(\frac{\partial Z}{\partial y}\right),$$

unde deducitur sequens Theorema 2:

Si fuerit $\int V\partial x=Z$, *tum semper erit*

$$\int\partial x\left(\frac{\partial V}{\partial y}\right)=\left(\frac{\partial Z}{\partial y}\right),\quad \textit{sive}\quad \partial x\left(\frac{\partial V}{\partial y}\right)=\partial\cdot\left(\frac{\partial Z}{\partial y}\right).$$

7. Progrediamur autem ulterius, et differentiale ipsius V, ex sola variabilitate ipsius p oriundum, contemplemur, ac reperiemus

$$\left(\frac{\partial V}{\partial p}\right)=\left(\frac{\partial\partial Z}{\partial x\partial p}\right)+p\left(\frac{\partial\partial Z}{\partial y\partial p}\right)+q\left(\frac{\partial\partial Z}{\partial p^2}\right)+r\left(\frac{\partial\partial Z}{\partial q\partial p}\right)+\text{etc.}+\left(\frac{\partial Z}{\partial y}\right).$$

Hinc iam si ponamus $\left(\frac{\partial Z}{\partial p}\right)=T$, erit

$$\left(\frac{\partial V}{\partial p}\right)=\left(\frac{\partial T}{\partial x}\right)+p\left(\frac{\partial T}{\partial y}\right)+q\left(\frac{\partial T}{\partial p}\right)+r\left(\frac{\partial T}{\partial q}\right)+\text{etc.}+\left(\frac{\partial Z}{\partial y}\right),$$

unde ergo sequitur fore

$$\partial x\left(\frac{\partial V}{\partial p}\right)=\partial T+\partial x\left(\frac{\partial Z}{\partial y}\right),$$

quod nobis suppeditat istud Theorema 3:

Si fuerit $\int V\partial x=Z$, *tum etiam semper erit*

$$\int\partial x\left(\frac{\partial V}{\partial p}\right)=\left(\frac{\partial Z}{\partial p}\right)+\int\partial x\left(\frac{\partial Z}{\partial y}\right),\quad \textit{sive}\quad \partial x\left(\frac{\partial V}{\partial p}\right)-\partial x\left(\frac{\partial Z}{\partial y}\right)=\partial\cdot\left(\frac{\partial Z}{\partial p}\right).$$

8. Sumta nunc sola quantitate q pro variabili simili modo orietur

$$\left(\frac{\partial V}{\partial q}\right)=\left(\frac{\partial\partial Z}{\partial x\partial q}\right)+p\left(\frac{\partial\partial Z}{\partial y\partial q}\right)+q\left(\frac{\partial\partial Z}{\partial p\partial q}\right)+r\left(\frac{\partial\partial Z}{\partial q^2}\right)+\text{etc.}+\left(\frac{\partial Z}{\partial p}\right),$$

unde si hic ponatur $\left(\frac{\partial Z}{\partial q}\right)=T$, erit

$$\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{ etc. } + \left(\frac{\partial Z}{\partial p}\right),$$

sicque per ∂x multiplicando fiet

$$\partial x\left(\frac{\partial V}{\partial q}\right) = \partial T + \partial x\left(\frac{\partial Z}{\partial p}\right).$$

Hinc orietur istud Theorema 4:

Si fuerit $\int V\partial x = Z$, *tum semper erit*

$$\int\partial x\left(\frac{\partial V}{\partial q}\right) = \left(\frac{\partial Z}{\partial q}\right) + \int\partial x\left(\frac{\partial Z}{\partial p}\right), \quad \textit{sive} \quad \partial x\left(\frac{\partial V}{\partial q}\right) - \partial x\left(\frac{\partial Z}{\partial p}\right) = \partial\cdot\left(\frac{\partial Z}{\partial q}\right).$$

9. Sumatur iam sola quantitas r pro variabili ac prodibit

$$\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial\partial Z}{\partial x\,\partial r}\right) + p\left(\frac{\partial\partial Z}{\partial y\,\partial r}\right) + q\left(\frac{\partial\partial Z}{\partial p\,\partial r}\right) + r\left(\frac{\partial\partial Z}{\partial q\,\partial r}\right) + \text{ etc. } + \left(\frac{\partial Z}{\partial q}\right),$$

unde si ponatur $\left(\frac{\partial Z}{\partial r}\right) = T$, erit

$$\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial Z}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{ etc. } + \left(\frac{\partial Z}{\partial q}\right).$$

Hinc igitur, ut supra patet, fore

$$\partial x\left(\frac{\partial V}{\partial r}\right) = \partial T + \partial x\left(\frac{\partial Z}{\partial q}\right),$$

sicque orietur sequens Theorema 5:

Si fuerit $\int V\partial x = Z$, *tum etiam semper erit*

$$\int\partial x\left(\frac{\partial V}{\partial r}\right) = \left(\frac{\partial Z}{\partial r}\right) + \int\partial x\left(\frac{\partial Z}{\partial q}\right), \quad \textit{sive} \quad \partial x\left(\frac{\partial V}{\partial r}\right) - \partial x\left(\frac{\partial Z}{\partial q}\right) = \partial\cdot\left(\frac{\partial Z}{\partial r}\right).$$

10. Haec iam ita sunt manifesta, ut superfluum foret ista theoremata ulterius prosequi. Ante autem quam repetitas differentiationes prosequamur, haec theoremata nobis inservire possunt, ad criterium illud generale demon-

strandum, quo primus ostendi formulam $\int V\partial x$ semper admittere integrationem, quoties fuerit

$$0=\left(\frac{\partial V}{\partial y}\right)-\frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial p}\right)+\frac{1}{\partial x^2}\partial\partial\cdot\left(\frac{\partial V}{\partial q}\right)-\frac{1}{\partial x^3}\partial^3\cdot\left(\frac{\partial V}{\partial r}\right)+\frac{1}{\partial x^4}\partial^4\cdot\left(\frac{\partial V}{\partial s}\right)-\text{etc.}$$

11. Ad hanc autem regulam demonstrandam, posito $\int V\partial x = Z$, per gradus progrediamur, prouti functio Z continuo plures continet litterarum x, y, p, q, r etc. Ac primo quidem contineat functio Z tantum binas variabiles x et y, exclusis omnibus differentialibus, ita ut sit

$$\left(\frac{\partial Z}{\partial p}\right)=0,\quad \left(\frac{\partial Z}{\partial q}\right)=0,\quad \left(\frac{\partial Z}{\partial r}\right)=0 \quad \text{etc.}$$

Hinc iam ex theoremate tertio erit $\left(\frac{\partial V}{\partial p}\right)=\left(\frac{\partial Z}{\partial y}\right)$, theorema autem secundum nobis praebet

$$\partial x\left(\frac{\partial V}{\partial y}\right)=\partial\cdot\left(\frac{\partial Z}{\partial y}\right).$$

Cum igitur inde sit $\left(\frac{\partial Z}{\partial y}\right)=\left(\frac{\partial V}{\partial p}\right)$, hoc valore substituto fiet

$$\partial x\left(\frac{\partial V}{\partial y}\right)=\partial\cdot\left(\frac{\partial V}{\partial p}\right),$$

ideoque per ∂x dividendo orietur haec aequatio:

$$0=\left(\frac{\partial V}{\partial y}\right)-\frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial p}\right),$$

prorsus ut criterium meum postulat. Quoniam enim ex Z litterae p, q, r etc. excluduntur, ob $\partial Z = V\partial x$ functio V neque litteram q, neque r, neque s etc. continere potest, unde etiam formulae $\left(\frac{\partial V}{\partial q}\right)$, $\left(\frac{\partial V}{\partial r}\right)$ etc. evanescunt.

12. Contineat nunc functio Z, praeter litteras x et y, etiam p, unde ob $\partial Z = V\partial x$ et $\partial p = q\partial x$, quantitas V etiamnunc q involvet, sequentes vero litterae r, s, t etc. excludentur. Cum igitur iam sit $\left(\frac{\partial Z}{\partial q}\right)=0$ multoque magis $\left(\frac{\partial Z}{\partial r}\right)=0$, $\left(\frac{\partial Z}{\partial s}\right)=0$ etc., theorema quartum nobis praebebit

$$\partial x\left(\frac{\partial V}{\partial q}\right)-\partial x\left(\frac{\partial Z}{\partial p}\right)=0,$$

unde fit

$$\left(\frac{\partial Z}{\partial p}\right)=\left(\frac{\partial V}{\partial q}\right),$$

qui valor in tertio theoremate substitutus dat

$$\partial x\left(\frac{\partial V}{\partial p}\right)-\partial x\left(\frac{\partial Z}{\partial y}\right)=\partial\cdot\left(\frac{\partial V}{\partial q}\right),$$

unde ergo erit

$$\left(\frac{\partial Z}{\partial y}\right)=\left(\frac{\partial V}{\partial p}\right)-\frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial q}\right),$$

qui valor in theoremate secundo substitutus praebet

$$\partial x\left(\frac{\partial V}{\partial y}\right)=\partial\cdot\left(\frac{\partial V}{\partial p}\right)-\frac{1}{\partial x}\partial\partial\cdot\left(\frac{\partial V}{\partial q}\right),$$

unde sequitur, prorsus ut nostrum criterium postulat,

$$0=\left(\frac{\partial V}{\partial y}\right)-\frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial p}\right)+\frac{1}{\partial x^2}\partial\partial\cdot\left(\frac{\partial V}{\partial q}\right).$$

13. Involvat nunc functio Z etiam litteram q, et quantitas V etiamnunc continebit litteram r, ob $\partial q=r\,\partial x$; sequentes vero inde excludentur. Cum igitur sit $\left(\frac{\partial Z}{\partial r}\right)=0$, theorema quintum nobis praebet

$$\partial x\left(\frac{\partial V}{\partial r}\right)-\partial x\left(\frac{\partial Z}{\partial q}\right)=0,$$

unde fit $\left(\frac{\partial Z}{\partial q}\right)=\left(\frac{\partial V}{\partial r}\right)$, qui valor in theoremate quarto substitutus suppeditat hanc aequationem:

$$\partial x\left(\frac{\partial V}{\partial q}\right)-\partial x\left(\frac{\partial Z}{\partial p}\right)=\partial\cdot\left(\frac{\partial V}{\partial r}\right),$$

unde colligitur

$$\left(\frac{\partial Z}{\partial p}\right)=\left(\frac{\partial V}{\partial q}\right)-\frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial r}\right).$$

Substituatur hic valor in theoremate tertio, fietque

$$\partial x\left(\frac{\partial V}{\partial p}\right) - \partial x\left(\frac{\partial Z}{\partial y}\right) = \partial\cdot\left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x}\partial\partial\cdot\left(\frac{\partial V}{\partial r}\right),$$

unde fit

$$\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial q}\right) + \frac{1}{\partial x^2}\partial^2\cdot\left(\frac{\partial V}{\partial r}\right),$$

qui valor in secundo theoremate substitutus praebet hanc aequationem:

$$0 = \left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial\cdot\left(\frac{\partial V}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial\cdot\left(\frac{\partial V}{\partial q}\right) - \frac{1}{\partial x^3}\partial^3\cdot\left(\frac{\partial V}{\partial r}\right).$$

14. Hoc igitur modo criterium supra memoratum, quod primum ex contemplatione maximorum et minimorum, via maxime indirecta, concluseram[1]), omni rigore est demonstratum; atque haec demonstratio non multum discrepat ab ea, quam sagacissimus noster Professor LEXELL exhibuit (Novor. Commentar. Acad. Scientiar. Petropol. Tomo XV. pag. 127). Nunc igitur formulas differentiales supra ex functione Z deductas per ulteriores differentiationes evolvamus, quandoquidem hinc innumerabilia alia theoremata, iis quae dedimus similia, derivari possunt.

EVOLUTIO FORMULAE

$$\left(\frac{\partial V}{\partial x}\right) = \left(\frac{\partial\partial Z}{\partial x^2}\right) + p\left(\frac{\partial\partial Z}{\partial y\partial x}\right) + q\left(\frac{\partial\partial Z}{\partial p\partial x}\right) + r\left(\frac{\partial\partial Z}{\partial q\partial x}\right) + \text{etc.}$$

PER ULTERIOREM DIFFERENTIATIONEM

15. Sumamus primo solam x pro variabili, ac facta differentiatione prodibit

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial^3 Z}{\partial x^3}\right) + p\left(\frac{\partial^3 Z}{\partial y\partial x^2}\right) + q\left(\frac{\partial^3 Z}{\partial p\partial x^2}\right) + r\left(\frac{\partial^3 Z}{\partial q\partial x^2}\right) + \text{etc.},$$

ubi, si statuamus $\left(\frac{\partial\partial Z}{\partial x^2}\right) = T$, erit

$$\left(\frac{\partial\partial V}{\partial x^2}\right) = \left(\frac{\partial T}{\partial x}\right) + p\left(\frac{\partial T}{\partial y}\right) + q\left(\frac{\partial T}{\partial p}\right) + r\left(\frac{\partial T}{\partial q}\right) + \text{etc.},$$

unde manifesto erit

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1) Vide EULERI *Institutiones calculi integralis* vol. III, § 91—97 appendicis de calculo variationum. *LEONHARDI EULERI Opera omnia*, series I, vol. 13, p. 408. H. D.

$$\partial x\left(\frac{\partial\partial V}{\partial x^2}\right)=\partial T=\partial\cdot\left(\frac{\partial\partial Z}{\partial x^2}\right),$$

atque hinc nascitur sequens theorema 6:

Si fuerit $\int V\partial x=Z$, *tum semper erit*

$$\int\partial x\left(\frac{\partial\partial V}{\partial x^2}\right)=\left(\frac{\partial\partial Z}{\partial x^2}\right),\quad \textit{sive}\quad \partial x\left(\frac{\partial\partial V}{\partial x^2}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial x^2}\right).$$

16. Sumatur nunc pro eadem formula sola y pro variabili, ac reperietur

$$\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\left(\frac{\partial^3 Z}{\partial y\,\partial x^2}\right)+p\left(\frac{\partial^3 Z}{\partial y^2\partial x}\right)+q\left(\frac{\partial^3 Z}{\partial p\,\partial x\,\partial y}\right)+r\left(\frac{\partial^3 Z}{\partial q\,\partial x\,\partial y}\right)+\text{ etc.},$$

ubi si statuamus $\left(\frac{\partial\partial Z}{\partial x\partial y}\right)=T$, erit

$$\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\left(\frac{\partial T}{\partial x}\right)+p\left(\frac{\partial T}{\partial y}\right)+q\left(\frac{\partial T}{\partial p}\right)+r\left(\frac{\partial T}{\partial q}\right)+\text{ etc.},$$

unde manifesto erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\partial T=\partial\cdot\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

sicque adepti sumus sequens theorema 7:

Si fuerit $\int V\partial x=Z$, *tum semper erit*

$$\int\partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),\quad \textit{sive}\quad \partial x\left(\frac{\partial\partial V}{\partial x\,\partial y}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right).$$

17. At si sola p variabilis capiatur, tum erit

$$\left(\frac{\partial\partial V}{\partial x\partial p}\right)=\left(\frac{\partial^3 Z}{\partial x^2\partial p}\right)+p\left(\frac{\partial^3 Z}{\partial x\partial y\partial p}\right)+q\left(\frac{\partial^3 Z}{\partial x\,\partial p^2}\right)+r\left(\frac{\partial^3 Z}{\partial q\,\partial x\partial p}\right)+\text{ etc. }+\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right).$$

Hinc ergo si ponatur $\left(\frac{\partial\partial Z}{\partial x\partial p}\right)=T$, erit

$$\partial x\left(\frac{\partial\partial V}{\partial x\partial p}\right)=\partial T+\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

hincque formatur sequens theorema 8:

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial\partial V}{\partial x \partial p}\right) = \left(\frac{\partial\partial Z}{\partial x \partial p}\right) + \int \partial x \left(\frac{\partial\partial Z}{\partial x \partial y}\right), \quad \textit{sive} \quad \partial x \left(\frac{\partial\partial V}{\partial x \partial p}\right) - \partial x \left(\frac{\partial\partial Z}{\partial x \partial y}\right) = \partial \cdot \left(\frac{\partial\partial Z}{\partial x \partial p}\right).$$

18. Sit nunc sola littera q variabilis, eritque

$$\left(\frac{\partial\partial V}{\partial x \partial q}\right) = \left(\frac{\partial^3 Z}{\partial x^2 \partial q}\right) + p\left(\frac{\partial^3 Z}{\partial x \partial y \partial q}\right) + q\left(\frac{\partial^3 Z}{\partial x \partial p \partial q}\right) + r\left(\frac{\partial^3 Z}{\partial x \partial q^2}\right) + \text{etc.} + \left(\frac{\partial\partial Z}{\partial p \partial x}\right),$$

hinc ergo, si ponatur $\left(\frac{\partial\partial Z}{\partial x \partial q}\right) = T$, erit

$$\partial x \left(\frac{\partial\partial V}{\partial x \partial q}\right) = \partial T + \partial x \left(\frac{\partial\partial Z}{\partial p \partial x}\right),$$

hincque formatur sequens theorema 9:

Si fuerit $\int V \partial x = Z$, *tum semper erit*

$$\int \partial x \left(\frac{\partial\partial V}{\partial x \partial q}\right) = \left(\frac{\partial\partial Z}{\partial x \partial q}\right) + \int \partial x \left(\frac{\partial\partial Z}{\partial p \partial x}\right), \quad \textit{sive} \quad \partial x \left(\frac{\partial\partial V}{\partial x \partial q}\right) - \partial x \left(\frac{\partial\partial Z}{\partial p \partial x}\right) = \partial \cdot \left(\frac{\partial\partial Z}{\partial x \partial q}\right).$$

EVOLUTIO FORMULAE

$$\left(\frac{\partial V}{\partial y}\right) = \left(\frac{\partial\partial Z}{\partial x \partial y}\right) + p\left(\frac{\partial\partial Z}{\partial y^2}\right) + q\left(\frac{\partial\partial Z}{\partial p \partial y}\right) + r\left(\frac{\partial\partial Z}{\partial q \partial y}\right) + \text{etc.}$$

PER ULTERIOREM DIFFERENTIATIONEM

19. Hanc evolutionem iam multo concinnius absolvere licebit. Cum enim forma proposita ita repraesentari possit, ut fit

$$\partial x \left(\frac{\partial V}{\partial y}\right) = \partial \cdot \left(\frac{\partial Z}{\partial y}\right),$$

singulas differentiationes in hac forma instituere poterimus. Ita si sola x variabilis sumatur, erit

$$\partial x \left(\frac{\partial\partial V}{\partial x \partial y}\right) = \partial \cdot \left(\frac{\partial\partial Z}{\partial x \partial y}\right),$$

quod iam est theorema 7 praecedentis evolutionis. Simili modo si sola y variabilis sumatur, prodibit

$$\partial x\left(\frac{\partial\partial V}{\partial y^2}\right) = \partial\cdot\left(\frac{\partial\partial Z}{\partial y^2}\right),$$

quod est novum theorema ad hanc evolutionem pertinens, unde fit

$$\int\partial x\left(\frac{\partial\partial V}{\partial y^2}\right) = \left(\frac{\partial\partial Z}{\partial y^2}\right).$$

Hinc patet si fuerit $\int V\partial x = Z$, tum semper fore

$$\int\partial x\left(\frac{\partial\partial V}{\partial y^2}\right) = \left(\frac{\partial\partial Z}{\partial y^2}\right).$$

At si sola p variabilis accipiatur, tum quadam circumspectione opus est, quoniam hoc casu non erit

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial p}\right) = \partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right),$$

sed insuper aliquod membrum accedet. Quoniam enim formula $\partial\cdot\left(\frac{\partial Z}{\partial y}\right)$ evoluta continet partem $p\,\partial x\left(\frac{\partial\partial Z}{\partial y^2}\right)$, huius differentiatio praebet $\partial x\left(\frac{\partial\partial Z}{\partial y^2}\right)$, quod ergo insuper adiici oportet, ita ut sit

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial p}\right) = \partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right) + \partial x\left(\frac{\partial\partial Z}{\partial y^2}\right),$$

consequenter sumtis integralibus erit

$$\int\partial x\left(\frac{\partial\partial V}{\partial y\,\partial p}\right) = \left(\frac{\partial\partial Z}{\partial y\,\partial p}\right) + \int\partial x\left(\frac{\partial\partial Z}{\partial y^2}\right),$$

sicque integratio formulae $\int\partial x\left(\frac{\partial\partial V}{\partial y\,\partial p}\right)$ insuper involvit formulam integralem $\int\partial x\left(\frac{\partial\partial Z}{\partial y^2}\right)$.

20. Sumatur nunc sola q pro variabili, et quia formula $\partial\cdot\left(\frac{\partial Z}{\partial y}\right)$ continet terminum $q\,\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right)$, variabilitas ipsius q producet terminum $\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right)$, sicque orietur ista aequatio:

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right).$$

Eodem modo patet, si sola littera r variabilis accipiatur, tum fore

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial r}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right).$$

Ac si sola s variabilis accipiatur, tum erit

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial s}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial r}\right),$$

sicque porro.

EVOLUTIO FORMULAE

$$\left(\frac{\partial V}{\partial p}\right)=\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right)+p\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right)+q\left(\frac{\partial\partial Z}{\partial p^2}\right)+r\left(\frac{\partial\partial Z}{\partial q\,\partial p}\right)+\text{etc.}+\left(\frac{\partial Z}{\partial y}\right)$$

QUAE REDUCITUR AD HANC FORMAM:

$$\partial x\left(\frac{\partial V}{\partial p}\right)=\partial\cdot\left(\frac{\partial Z}{\partial p}\right)+\partial x\left(\frac{\partial Z}{\partial y}\right).$$

21. Quodsi hic vel sola x vel sola y variabilis accipiatur, haec forma simpliciter differentiata ad quaesitum perducit: priori scilicet casu prodit

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial p}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right)+\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial y}\right),$$

posteriore vero erit

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial p}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y^2}\right),$$

haecque duae formulae iam ante prodierunt.

22. Sin autem littera p variabilis statuatur, quoniam formula $\partial\cdot\left(\frac{\partial Z}{\partial p}\right)$ continet partem $p\,\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right)$, huius differentiatio praebet $\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right)$, quod ergo ad differentialia, ex reliquis membris oriunda, insuper addi debet; hoc ergo modo prodibit ista aequalitas:

$$\partial x\left(\frac{\partial\partial V}{\partial p^2}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial p^2}\right)+2\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right),$$

unde intelligitur ob ∂p^2, quod in prima formula occurrit, postremum terminum duplicari debere. Theorematibus autem hinc deducendis non immoramur, quandoquidem deinceps theoremata multo generaliora producere licebit.

23. Sumamus nunc solam q variabilem, et quoniam in formula $\partial\cdot\left(\frac{\partial Z}{\partial p}\right)$ evoluta occurrit terminus $q\,\partial x\left(\frac{\partial\partial Z}{\partial p^2}\right)$, ex hoc per differentiationem nascitur terminus $\partial x\left(\frac{\partial\partial Z}{\partial p^2}\right)$. Hinc ergo facta tota differentiatione perveniemus ad hanc aequationem:

$$\partial x\left(\frac{\partial\partial V}{\partial p\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p^2}\right),$$

ubi patet, ob bina elementa ∂p et ∂q, insuper duos terminos adiici oportere, id quod etiam eveniet, si sola r pro variabili sumatur; nam quia formula $\partial\cdot\left(\frac{\partial Z}{\partial p}\right)$ continet terminum $r\,\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right)$, ex hoc per differentiationem prodibit $\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right)$, quem ad reliquas partes insuper adiici oportet; hocque modo impetrabimus hanc aequationem:

$$\partial x\left(\frac{\partial\partial V}{\partial p\,\partial r}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial p\,\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right),$$

ubi iterum ob elementa ∂p et ∂r duo membra accesserunt.

EVOLUTIO FORMULAE

$$\left(\frac{\partial V}{\partial q}\right)=\left(\frac{\partial\partial Z}{\partial x\,\partial q}\right)+p\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right)+q\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right)+r\left(\frac{\partial\partial Z}{\partial q^2}\right)+\text{etc.}+\left(\frac{\partial Z}{\partial p}\right),$$

QUAE REDUCTA EST AD HANC:

$$\partial x\left(\frac{\partial V}{\partial q}\right)=\partial\cdot\left(\frac{\partial Z}{\partial q}\right)+\partial x\left(\frac{\partial Z}{\partial p}\right).$$

24. Si hic vel x vel y solum variabilis capiatur, nihil in differentiatione de novo accedit, eritque casu priore

$$\partial x\left(\frac{\partial\partial V}{\partial x\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial x\,\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial x\,\partial p}\right),$$

posteriore vero casu

$$\partial x\left(\frac{\partial\partial V}{\partial y\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right).$$

In reliquis autem differentiationibus elementum ∂p suppeditat, praeter differentiationem solitam, insuper membrum $\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right)$, at vero elementum ∂q producit $\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right)$; elementum porro ∂r producit $\partial x\left(\frac{\partial\partial Z}{\partial q^2}\right)$, elementum ∂s vero praebet $\partial x\left(\frac{\partial\partial Z}{\partial q\,\partial r}\right)$ etc., quibus observatis obtinebuntur sequentes aequationes:

$$\text{I.}\quad \partial x\left(\frac{\partial\partial V}{\partial p\,\partial q}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p^2}\right)+\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial q}\right),$$

$$\text{II.}\quad \partial x\left(\frac{\partial\partial V}{\partial q^2}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial q^2}\right)+2\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial q}\right),$$

$$\text{III.}\quad \partial x\left(\frac{\partial\partial V}{\partial q\,\partial r}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial q\,\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial r}\right)+\partial x\left(\frac{\partial\partial Z}{\partial q^2}\right),$$

$$\text{IV.}\quad \partial x\left(\frac{\partial\partial V}{\partial q\,\partial s}\right)=\partial\cdot\left(\frac{\partial\partial Z}{\partial q\,\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial p\,\partial s}\right)+\partial x\left(\frac{\partial\partial Z}{\partial q\,\partial r}\right).$$

25. Ex his iam abunde perspicitur, perpetuo, quoties vel sola x, vel sola y variabilis accipitur, differentiationem more consueto institui debere, nihilque insuper esse adiiciendum; sin autem reliquae litterae p, q, r, s etc. variabiles accipiantur, tum pro quolibet elemento sive ∂p, sive ∂q, sive ∂r etc. praeterea unum novum terminum accedere debere. Hinc igitur pro solis elementis ∂x et ∂y iam sequens theorema latissime patens constitui potest:

THEOREMA GENERALE 1

Si fuerit $\int V\partial x = Z$, tum semper erit

$$\int\partial x\left(\frac{\partial^{\alpha+\beta}V}{\partial x^{\alpha}\,\partial y^{\beta}}\right)=\left(\frac{\partial^{\alpha+\beta}Z}{\partial x^{\alpha}\,\partial y^{\beta}}\right).$$

EVOLUTIO HARUM FORMULARUM SI SOLA p PRO VARIABILI ACCIPIATUR

26. Quemadmodum iam vidimus, cum sit

$$\partial x\left(\frac{\partial V}{\partial p}\right)=\partial\cdot\left(\frac{\partial Z}{\partial p}\right)+\partial x\left(\frac{\partial Z}{\partial y}\right),$$

tum fore

$$\partial x\left(\frac{\partial\partial V}{\partial p^2}\right) = \partial\cdot\left(\frac{\partial\partial Z}{\partial p^2}\right) + 2\partial x\left(\frac{\partial\partial Z}{\partial y\,\partial p}\right),$$

ita si porro differentiemus, ex sola variabilitate ipsius p prodibit:

$$\text{I}^0.\ \partial x\left(\frac{\partial^3 V}{\partial p^3}\right) = \partial\cdot\left(\frac{\partial^3 Z}{\partial p^3}\right) + 3\partial x\left(\frac{\partial^3 Z}{\partial y\,\partial p^2}\right),$$

$$\text{II}^0.\ \partial x\left(\frac{\partial^4 V}{\partial p^4}\right) = \partial\cdot\left(\frac{\partial^4 Z}{\partial p^4}\right) + 4\partial x\left(\frac{\partial^4 Z}{\partial y\,\partial p^3}\right),$$

$$\text{III}^0.\ \partial x\left(\frac{\partial^5 V}{\partial p^5}\right) = \partial\cdot\left(\frac{\partial^5 Z}{\partial p^5}\right) + 5\partial x\left(\frac{\partial^5 Z}{\partial y\,\partial p^4}\right),$$

unde generaliter habebimus

$$\partial x\left(\frac{\partial^\gamma V}{\partial p^\gamma}\right) = \partial\cdot\left(\frac{\partial^\gamma Z}{\partial p^\gamma}\right) + \gamma\partial x\left(\frac{\partial^\gamma Z}{\partial y\,\partial p^{\gamma-1}}\right),$$

hincque deducimus sequens

THEOREMA GENERALE 2

Si fuerit $\int V\partial x = Z$, tum semper erit

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma} V}{\partial x^\alpha\,\partial y^\beta\,\partial p^\gamma}\right) = \left(\frac{\partial^{\alpha+\beta+\gamma} Z}{\partial x^\alpha\,\partial y^\beta\,\partial p^\gamma}\right) + \gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma} Z}{\partial x^\alpha\,\partial y^{\beta+1}\,\partial p^{\gamma-1}}\right).$$

27. Quodsi iam ulterius quantitas q pro variabili sumatur, et differentiatio continuo repetatur, investigationem sequenti modo suscipiamus. Quoniam elementa ∂x et ∂y nihil turbant, proficiscamur a formula supra inventa:

$$\partial x\left(\frac{\partial^\gamma V}{\partial p^\gamma}\right) = \partial\cdot\left(\frac{\partial^\gamma Z}{\partial p^\gamma}\right) + \gamma\partial x\left(\frac{\partial^\gamma Z}{\partial y\,\partial p^{\gamma-1}}\right),$$

unde variabilitas solius q primo dabit

$$\partial x\left(\frac{\partial^{\gamma+1} V}{\partial p^\gamma\,\partial q}\right) = \partial\cdot\left(\frac{\partial^{\gamma+1} Z}{\partial p^\gamma\,\partial q}\right) + \gamma\partial x\left(\frac{\partial^{\gamma+1} Z}{\partial y\,\partial p^{\gamma-1}\,\partial q}\right) + \partial x\left(\frac{\partial^{\gamma+1} Z}{\partial p^{\gamma+1}}\right).$$

28. Quodsi iam hanc formam ulterius secundum ∂q differentiemus, perveniemus ad hanc aequationem:

$$\partial x\left(\frac{\partial^{\gamma+2}V}{\partial p^{\gamma}\,\partial q^{2}}\right)=\partial\cdot\left(\frac{\partial^{\gamma+2}Z}{\partial p^{\gamma}\,\partial q^{2}}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+2}Z}{\partial y\,\partial p^{\gamma-1}\partial q^{2}}\right)+2\,\partial x\left(\frac{\partial^{\gamma+2}Z}{\partial p^{\gamma+1}\partial q}\right),$$

et denuo differentiando prodibit

$$\partial x\left(\frac{\partial^{\gamma+3}V}{\partial p^{\gamma}\,\partial q^{3}}\right)=\partial\cdot\left(\frac{\partial^{\gamma+3}Z}{\partial p^{\gamma}\,\partial q^{3}}\right)+\gamma\,\partial x\left(\frac{\partial^{\gamma+3}Z}{\partial y\,\partial p^{\gamma-1}\partial q^{3}}\right)+3\,\partial x\left(\frac{\partial^{\gamma+3}Z}{\partial p^{\gamma+1}\partial q^{2}}\right),$$

haecque sufficiunt ad constituendum sequens

THEOREMA GENERALE 3

Si fuerit $\int V\partial x=Z$, tum semper erit:

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}V}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}}\right)=\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}}\right)+\gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}Z}{\partial x^{\alpha}\partial y^{\beta+1}\partial p^{\gamma-1}\partial q^{\delta}}\right)+\delta\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma+1}\partial q^{\delta-1}}\right).$$

29. Iam pluribus ambagibus opus non erit ad sequens theorema generalissimum constituendum:

THEOREMA GENERALISSIMUM

Si fuerit $\int V\partial x=Z$, tum semper erit

$$\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta}V}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}\partial r^{\varepsilon}\partial s^{\zeta}}\right)=\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}\partial r^{\varepsilon}\partial s^{\zeta}}\right)+\gamma\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta}Z}{\partial x^{\alpha}\partial y^{\beta+1}\partial p^{\gamma-1}\partial q^{\delta}\partial r^{\varepsilon}\partial s^{\zeta}}\right)$$

$$+\delta\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma+1}\partial q^{\delta-1}\partial r^{\varepsilon}\partial s^{\zeta}}\right)+\varepsilon\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta+1}\partial r^{\varepsilon-1}\partial s^{\zeta}}\right)+\zeta\int\partial x\left(\frac{\partial^{\alpha+\beta+\gamma+\delta+\varepsilon+\zeta}Z}{\partial x^{\alpha}\partial y^{\beta}\partial p^{\gamma}\partial q^{\delta}\partial r^{\varepsilon+1}\partial s^{\zeta-1}}\right).$$

DE FORMULIS DIFFERENTIALIBUS SECUNDI GRADUS QUAE INTEGRATIONEM ADMITTUNT

Conventui exhibita die 24. Aprilis 1777
Commentatio 700 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae 11 (1793), 1798, p. 3—26
Summarium ibidem p. 159—161

SUMMARIUM

Le but de l'Auteur de ce Mémoire a été d'examiner, quelles sont les valeurs qu'on peut donner à V, fonction de x, y et $p = \frac{\partial y}{\partial x}$, pour que la formule $V\partial p$ devienne intégrable. On connoît depuis longtems les critères de l'intégrabilité de la formule $\int Z\partial x$, Z étant fonction de x, y et p; et il est facile à voir qu'on peut aisement transformer la formule proposée $\int V\partial p$ en $\int Z\partial x$, et s'assurer par là si cette formule est intégrable ou non; mais l'équation de condition qui en résulte n'étant pas propre à faire connoître les valeurs de V, elle n'est d'aucun usage pour le problème en question; c'est pourquoi l'auteur examine dans une suite de problèmes les conditions qui rendent intégrables la formule proposée $\int V\partial p$ pour les cas suivans:

1°.) $V = Px + Qy$,
2°.) $V = \Pi(Mx + Ny)$,
3°.) $V = (px - y)^{n-1}(Px + Qy)$,
4°.) $V = (px - y)^{n-1}\Pi(Mx + Ny)$,

P, Q, M, N, Π étant fonctions de $p = \frac{\partial y}{\partial x}$.

Pour donner une idée de la méthode, nous transcrirons ici la solution la plus simple du premier problème que M. EULER a traité de deux manières différentes. La voici:

Comme $x\partial p = \partial\cdot(px - y)$, on aura

$$\int Px\partial p = P(px - y) - \int(px - y)\,\partial P\ ;$$

de même à cause de

$$\frac{y\partial p}{pp} = \partial\cdot\left(x - \frac{y}{p}\right),$$

on aura

$$\int Q y \partial p = Qpp\left(x - \frac{y}{p}\right) - \int\left(x - \frac{y}{p}\right)\partial\cdot Qpp\,.$$

De là il suit que

$$\int V\partial p = P(px - y) - \int(px - y)\partial P + Qpp\left(x - \frac{y}{p}\right) - \int\left(x - \frac{y}{p}\right)\partial\cdot Qpp\,.$$

Ainsi il faut que

$$\int(px - y)\partial P + \int\left(x - \frac{y}{p}\right)\partial\cdot Qpp = 0\,,$$

ou bien que

$$\partial P + p\partial Q + 2Q\partial p = 0\,,$$

équation de condition qui donne la relation entre les fonctions P et Q cherchée, telle que l'intégrabilité l'exige.

A la suite du premier, du second et du quatrième problème général se trouvent plusieurs applications à des formules différentielles particulières assez compliquées, dont les intégrales sont pour la plupart algébriques et d'une forme bien simple.

1. Inter tales formulas differentiales secundi gradus, quae integrationem admittunt, imprimis notatu digna est haec formula:

$$\frac{(x\partial x + y\partial y)(\partial y\partial\partial x - \partial x\partial\partial y)}{(\partial x^2 + \partial y^2)^{\frac{3}{2}}},$$

quae, si x et y designent coordinatas orthogonales lineae curvae, oritur, si elementum $x\partial x + y\partial y$ dividatur per radium osculi huius curvae; quandoquidem constat istius formulae integrale esse

$$\frac{y\partial x - x\partial y}{\sqrt{(\partial x^2 + \partial y^2)}},$$

quemadmodum calculum instituenti, dum huius formulae differentiale quaeritur, facile patebit. Cum igitur haec integratio neutiquam sit obvia, et plures ambages postulet, hoc argumentum hic accuratius pertractare constitui, unde intelligi poterit, quemadmodum plures aliae huiusmodi formulae inveniri queant, quae pariter integrationem admittant.

2. Quod quo facilius fieri possit, differentialia secundi gradus ex calculo eliminemus, quod commodissime fiet ponendo $\partial y = p\,\partial x$, ita ut loco differentialium secundorum in calculum introducatur ista nova quantitas $p = \frac{\partial y}{\partial x}$, quippe quae rationem differentialium primorum continet. Tum igitur erit

$$x\,\partial x + y\,\partial y = \partial x(x + py)$$

atque

$$\partial x^2 + \partial y^2 = \partial x^2(1 + pp),$$

ideoque denominator formulae propositae fit

$$(\partial x^2 + \partial y^2)^{\frac{3}{2}} = \partial x^3(1 + pp)^{\frac{3}{2}};$$

denique pro altero numeratoris factore habetur

$$\partial y\,\partial\partial x = p\,\partial x\,\partial\partial x,$$

et ob

$$\partial\partial y = p\,\partial\partial x + \partial p\,\partial x$$

erit

$$\partial x\,\partial\partial y = p\,\partial x\,\partial\partial x + \partial p\,\partial x^2,$$

sicque alter ille factor erit

$$\partial y\,\partial\partial x - \partial x\,\partial\partial y = -\,\partial p\,\partial x^2,$$

quibus substitutis formula proposita hanc induet formam:

$$-\frac{\partial p(x + py)}{(1 + pp)^{\frac{3}{2}}},$$

cuius ergo integrale erit

$$\frac{y\,\partial x - x\,\partial y}{\sqrt{(\partial x^2 + \partial y^2)}} = \frac{y - px}{\sqrt{(1 + pp)}},$$

quippe cuius differentiale superiorem praebet formulam.

3. Cum igitur facta hac substitutione in formulam differentio-differentialem unicum ingrediatur differentiale ∂p, sic in genere contemplabor hanc formulam: $V\partial p$, inquisiturus cuiusmodi valores isti litterae V tribui debeant,

ut formulae $V\partial p$ integrale exhiberi queat; ubi quidem evidens est, hanc quantitatem V certam esse oportere functionem trium variabilium x, y et p, quae ergo quomodo comparata esse debeat, ut integratio succedat, hic accuratius investigare constitui[1]).

4. Ac primo quidem ex iis, quae olim circa integrabilitatem formularum differentialium altiorum ordinum tradidi, criteria haud difficulter exhiberi poterunt, unde dignosci queat, utrum talis formula $V\partial p$ integrationem admittat nec ne. Tum temporis autem contemplatus sum talem formam $\int Z\,\partial x$, ubi positis

$$\partial y = p\,\partial x,\quad \partial p = q\,\partial x,\quad \partial q = r\,\partial x,\quad \partial r = s\,\partial x \quad \text{etc.}$$

littera Z denotabat functionem ex litteris x, y, p, q, r, s etc. utcunque compositam, atque ostendi[2]), quoties haec formula $\int Z\,\partial x$ fuerit integrabilis, tum semper fore

$$0 = \left(\frac{\partial Z}{\partial y}\right) - \frac{1}{\partial x}\partial\cdot\left(\frac{\partial Z}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial\cdot\left(\frac{\partial Z}{\partial q}\right) - \frac{1}{\partial x^3}\partial^3\cdot\left(\frac{\partial Z}{\partial r}\right) + \frac{1}{\partial x^4}\partial^4\cdot\left(\frac{\partial Z}{\partial s}\right) - \text{etc.}$$

Sin autem ista quantitas non sponte nihilo evadat aequalis, tum ista aequatio eam relationem inter x et y exprimit, pro qua formula integralis $\int Z\,\partial x$ maximum minimumve valorem nanciscatur.

5. Ut igitur formulam $\int V\partial p$, quam hic consideramus, ad istam formam: $\int Z\,\partial x$ reducamus, statuamus $\partial p = q\,\partial x$, ut formula nostra evadat $Vq\,\partial x$, ideoque $Z = Vq$; ubi notetur, quantitatem V tantum ternas litteras x, y et p complecti, quo observato erit

$$\left(\frac{\partial Z}{\partial y}\right) = \left(\frac{q\,\partial V}{\partial y}\right),$$

deinde

$$\left(\frac{\partial Z}{\partial p}\right) = \left(\frac{q\,\partial V}{\partial p}\right) \quad \text{et} \quad \left(\frac{\partial Z}{\partial q}\right) = V,$$

sicque criterium integrabilitatem indicans erit

$$0 = \left(\frac{q\,\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial\cdot\left(\frac{q\,\partial V}{\partial p}\right) + \frac{1}{\partial x^2}\partial\partial\cdot V,$$

1) Vide notam p. 94. H. D.
2) Vide notam p. 127. H. D.

quam aequationem etiam ita referre licet:

$$0 = q\left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial\cdot\left[q\left(\frac{\partial V}{\partial p}\right) - \frac{1}{\partial x}\partial V\right],$$

tum vero etiam, ob $q\,\partial x = \partial p$, hac ratione ea repraesentari potest:

$$0 = \partial p\left(\frac{\partial V}{\partial y}\right) - \frac{1}{\partial x}\partial\cdot\left[\partial p\left(\frac{\partial V}{\partial p}\right) - \partial V\right].$$

6. Cum igitur in genere per huiusmodi characteres iam satis usu receptos sit

$$\partial V = \partial x\left(\frac{\partial V}{\partial x}\right) + \partial y\left(\frac{\partial V}{\partial y}\right) + \partial p\left(\frac{\partial V}{\partial p}\right),$$

hoc valore substituto criterium desideratum hac exprimetur ratione:

$$0 = \partial p\left(\frac{\partial V}{\partial y}\right) + \frac{1}{\partial x}\partial\cdot\left[\partial x\left(\frac{\partial V}{\partial x}\right) + \partial y\left(\frac{\partial V}{\partial y}\right)\right],$$

quae ergo aequatio continet criterium desideratum; ita ut, quoties ista formula revera nihilo evadit aequalis, tum semper certi esse queamus, istam formulam propositam $V\partial p$ esse integrabilem.

7. Quoniam V per hypothesin est functio involvens has tres variabiles x, y et p, sit differentiationem more solito instituendo

$$\partial V = M\partial x + N\partial y + P\partial p$$

atque criterium continebitur in hac aequatione:

$$0 = N\partial p + \partial\cdot(M + Np),$$

quae porro evolvitur in hanc:

$$0 = 2N\partial p + \partial M + p\,\partial N.$$

Cuius vis quo clarius perspiciatur, applicemus istud criterium ad formulam initio propositam

$$\frac{\partial p(x+py)}{(1+pp)^{\frac{3}{2}}},$$

ubi, cum sit

$$V = \frac{x+py}{(1+pp)^{\frac{3}{2}}},$$

sumta sola x variabili, reperitur

$$M = \frac{1}{(1+pp)^{\frac{3}{2}}},$$

sumta autem sola y variabili, fiet

$$N = \frac{p}{(1+pp)^{\frac{3}{2}}};$$

hinc ergo erit

$$\partial M = -\frac{3p\partial p}{(1+pp)^{\frac{5}{2}}} \quad \text{et} \quad \partial N = \frac{(1-2pp)\partial p}{(1+pp)^{\frac{5}{2}}},$$

quibus valoribus substitutis, quia est

$$1. \qquad 2N\partial p = \frac{2p\partial p}{(1+pp)^{\frac{3}{2}}} = \frac{2p\partial p(1+pp)}{(1+pp)^{\frac{5}{2}}},$$

$$2. \qquad \partial M = -\frac{3p\partial p}{(1+pp)^{\frac{5}{2}}},$$

$$3. \qquad p\partial N = \frac{p\partial p(1-2pp)}{(1+pp)^{\frac{5}{2}}},$$

harum formularum summa manifesto ad nihilum redigitur. Ex quo intelligitur hanc formulam revera esse integrabilem, etiamsi integrale non constaret.

8. Quoniam autem hic nobis potius est propositum in valores idoneos pro littera V sumendos inquirere, quibus formula differentialis $V\partial p$ integrationem admittit, criterium inventum nullum usum praestare potest; quam ob rem investigationem nostram a casibus simplicissimis exordiamur, quibus formula nobis proposita integrationem admittit, inter quos sine dubio omnium simplicissimus est, quando V denotat quantitatem constantem. Sit igitur $V = 1$,

eritque $\int \partial p = p$. Hinc autem porro sequitur, si differentiale ∂p in functionem quamcunque istius integralis p, quae sit $\varDelta : p$, ducatur, tum semper hanc formulam $\partial p \varDelta : p$ fore integrabilem, quod quidem per se est perspicuum. Hic enim sub voce integrabilitatis non tantum intelligimus quicquid algebraice exhiberi poterit, sed in genere, quicquid per quantitates utcunque transcendentes assignari potest.

9. Secundus casus simplicissimus, quo formula $V \partial p$ integrabilis evadit, est quando $V = x$, ita ut formula differentialis sit $= x \partial p$. Quoniam enim per reductionem notissimam sit $\int x \partial p = px - \int p \partial x$, ob $p \partial x = \partial y$ erit hoc integrale $\int x \partial p = px - y$. Hinc igitur si $\varDelta : (px - y)$ denotet functionem quamcunque formulae $px - y$, semper quoque integrationem admittet haec formula differentialis multo latius patens: $x \partial p \varDelta : (px - y)$, quippe quae, posito $px - y = V$, ob $\partial V = x \partial p$ induit hanc formam: $\partial V \varDelta : V$.

10. Praeterea vero datur etiam tertius casus simplicissimus, quo formula nostra $V \partial p$ fit integrabilis, qui oritur ponendo $V = \frac{y}{pp}$. Per eandem enim reductionem, qua est $\int t \partial u = tu - \int u \partial t$, sumendo $t = y$ et $\partial u = \frac{\partial p}{pp}$, unde fit $\partial t = \partial y = p \partial x$ et $u = \frac{-1}{p}$, fiet

$$\int \frac{y \partial p}{pp} = \frac{-y}{p} + \int \partial x = x - \frac{y}{p}.$$

Si igitur porro $\varDelta : \left(x - \frac{y}{p}\right)$ denotet functionem quamcunque formulae $x - \frac{y}{p}$, etiam semper integrabilis erit haec formula differentialis multo generalior:

$$\frac{y \partial p}{pp} \varDelta : \left(x - \frac{y}{p}\right).$$

Quodsi enim ponatur $x - \frac{y}{p} = V$, ob $\partial V = \frac{y \partial p}{pp}$, haec forma evadit $= \partial V \varDelta : V$, quae manifesto semper est integrabilis.

11. His casibus principalibus constitutis inquiramus quoque in casus magis compositos, quibus formula generalis $V \partial p$ itidem fiet integrabilis, quem in finem sequentia problemata pertractemus.

PROBLEMA 1

Quaerantur duae functiones ipsius p, quae sint P et Q, ita comparatae, ut ista formula differentialis: $\partial p(Px + Qy)$ evadat integrabilis.

SOLUTIO

12. Quoniam haec formula duas involvit partes, eas per allatam reductionem seorsim evolvamus, ac primo quidem erit

$$\int Px\,\partial p = x\int P\partial p - \int\partial x\int P\partial p,$$

ubi quidem integrale $\int P\partial p$ ut quantitas cognita spectari potest, propterea quod P denotat functionem ipsius p. Simili modo pro altera parte erit

$$\int Qy\,\partial p = y\int Q\,\partial p - \int\partial y\int Q\,\partial p,$$

ubi postrema membra utrinque continent formulas per se non integrabiles, unde necesse est[1]), ut binis formulis in unam summam collectis haec duo membra postrema se mutuo tollant. Fiat igitur

$$\int\partial x\int P\partial p + \int\partial y\int Q\,\partial p = 0,$$

ideoque differentiando, ob $\partial y = p\partial x$, erit

$$\int P\partial p + p\int Q\,\partial p = 0.$$

Nunc denuo differentiemus atque obtinebimus

$$P + \int Q\,\partial p + Qp = 0,$$

quae iterum differentiata praebet

$$\partial P + p\,\partial Q + 2Q\,\partial p = 0$$[2]),

in qua aequatione relatio quaesita inter binas functiones P et Q continetur.

1) Quaedam in hac demonstratione deficiunt, sed quae EULERUS scripsit recta sunt. H. D.

2) Haec formula eadem est ac formula data § 7

$$0 = 2N dp + dM + p dN.$$ H. D.

13. Quodsi haec ultima aequatio ducatur in p, prodibit

$$p\partial P + \partial\cdot Qpp = 0;$$

unde patet, si altera harum duarum functionum P et Q fuerit cognita, hinc alteram determinari posse. Si enim verbi gratia data fuerit functio P, ob $\int p\partial P + Qpp = C$, erit

$$Q = \frac{C - \int p\partial P}{pp}.$$

Sin autem altera functio Q fuerit data, ex priore formula erit

$$\partial P = -p\partial Q - 2Q\partial p,$$

ideoque integrando

$$P = C - \int(p\partial Q + 2Q\partial p)$$

sive etiam

$$P = C - Qp - \int Q\partial p.$$

14. Quando vero istae duae functiones P et Q hoc modo rite fuerint determinatae, tum integrale formulae differentialis propositae $\partial p(Px + Qy)$ ita exprimetur, ut sit $= x\int P\partial p + y\int Q\partial p$. Atque iam notavimus, alterutram functionum P et Q pro lubitu assumi posse. Quin etiam certa quaedam relatio inter P et Q statui potest. Veluti si velimus ut sit $P = nQp$, hoc valore in aequatione differentiali substituto fiet

$$(n+2)Q\partial p + (n+1)p\partial Q = 0,$$

unde porro deducitur

$$\frac{(n+2)\partial p}{p} + \frac{(n+1)\partial Q}{Q} = 0,$$

cuius integrale est

$$(n+2)lp + (n+1)lQ = lC,$$

hincque porro $p^{n+2}Q^{n+1} = C$, ex quo deducitur

$$Q = \frac{C}{p^{\frac{n+2}{n+1}}}, \quad \text{consequenter} \quad P = \frac{nC}{p^{\frac{1}{n+1}}}.$$

15. Quoniam integrale inventum est $x\int P\partial p + y\int Q\,\partial p$, hae duae formulae integrales duas constantes accipere sunt censendae, ita ut integrale verum ita prodeat expressum: $x\int P\partial p + y\int Q\,\partial p + \alpha x + \beta y$, ubi constantes α et β quovis casu ita determinari oportet, ut sumtis differentialibus elementum ∂x ex calculo excedat, id quod fit si fuerit

$$\partial x \int P\partial p + p\,\partial x \int Q\,\partial p + \alpha\partial x + \beta p\,\partial x = 0,$$

unde prodit, uti iam invenimus,

$$P\partial p + \partial p\int Q\partial p + Qp\partial p + \beta\partial p = 0,$$

quae per ∂p divisa et denuo differentiata praebet

$$\partial P + 2Q\,\partial p + p\,\partial Q = 0,$$

quae aequatio exprimit relationem requisitam inter P et Q.

ALIA SOLUTIO EIUSDEM PROBLEMATIS

16. Cum sit $x\,\partial p$ differentiale formulae $px - y$, erit per reductionem

$$\int Px\,\partial p = P(px - y) - \int(px - y)\,\partial P;$$

deinde cum sit $\frac{y\partial p}{pp}$ differentiale formulae $x - \frac{y}{p}$, erit per reductionem:

$$\int Qy\partial p = \int Qpp\cdot\frac{y\partial p}{pp} = Qpp\left(x - \frac{y}{p}\right) - \int\left(x - \frac{y}{p}\right)\partial\cdot Qpp.$$

His igitur coniungendis integrale formulae propositae erit

$$P(px - y) + Qpp\left(x - \frac{y}{p}\right) - \int(px - y)\partial P - \int\left(x - \frac{y}{p}\right)\partial\cdot Qpp,$$

unde evidens est partes postremas integrales nihilo aequales fieri debere. Hinc sumtis differentialibus statui debet

$$(px - y)\partial P + \left(x - \frac{y}{p}\right)\partial\cdot Qpp = 0,$$

quae aequatio per $px - y$ divisa dat

$$\partial P + \frac{1}{p}\partial \cdot Qpp = 0,$$

sive

$$\partial P + p\partial Q + 2Q\partial p = 0,$$

quae est eadem aequatio inter P et Q, quam prior solutio suppeditavit.

17. Quoniam supra vidimus hanc formulam

$$\frac{(x + py)\,\partial p}{(1 + pp)^{\frac{3}{2}}}$$

integrationem admittere, facta applicatione hic erit

$$P = \frac{1}{(1 + pp)^{\frac{3}{2}}} \quad \text{et} \quad Q = \frac{p}{(1 + pp)^{\frac{3}{2}}}.$$

Spectemus nunc quantitatem P tanquam cognitam et videamus an pro Q eundem valorem reperiamus. Cum igitur

$$\partial P = \frac{-3p\,\partial p}{(1 + pp)^{\frac{5}{2}}},$$

aequatio inventa evadet

$$\frac{-3p\partial p}{(1 + pp)^{\frac{5}{2}}} + p\partial Q + 2Q\partial p = 0,$$

quae ducta in p praebet

$$\partial \cdot Qpp = \frac{3pp\partial p}{(1 + pp)^{\frac{5}{2}}}, \quad \text{ideoque} \quad Qpp = \int \frac{3pp\partial p}{(1 + pp)^{\frac{5}{2}}}.$$

Levi autem attentione adhibita patebit esse

$$\int \frac{3pp\,\partial p}{(1 + pp)^{\frac{5}{2}}} = \frac{p^3}{(1 + pp)^{\frac{3}{2}}},$$

sicque erit

$$Qpp = \frac{p^3}{(1+pp)^{\frac{3}{2}}},$$

ideoque

$$Q = \frac{p}{(1+pp)^{\frac{3}{2}}} + \frac{C}{pp}.$$

18. Hinc igitur videmus pro valore

$$P = \frac{1}{(1+pp)^{\frac{3}{2}}}$$

non solum esse

$$Q = \frac{p}{(1+pp)^{\frac{3}{2}}},$$

sed generalius sumi posse

$$Q = \frac{p}{(1+pp)^{\frac{3}{2}}} + \frac{C}{pp},$$

ita ut iam haec formula differentiationem admittat. Cum igitur in genere integrale inventum sit

$$P(px-y) + Qpp\left(x - \frac{y}{p}\right),$$

his valoribus substitutis integrale erit

$$\frac{px-y}{(1+pp)^{\frac{3}{2}}} + \frac{pp(px-y)}{(1+pp)^{\frac{3}{2}}} + \frac{C(px-y)}{p},$$

quod reducitur ad hanc formam:

$$\frac{px-y}{\sqrt{(1+pp)}} + \frac{C(px-y)}{p}.$$

PROBLEMA 2

Si M et N fuerint functiones quaecunque datae ipsius p, invenire eiusdem functionem Π, ut ista formula differentialis: $(Mx+Ny)\Pi\partial p$, integrationem admittat.

SOLUTIO

19. Si hoc problema cum praecedente comparemus, facile patet functiones illas litteris P et Q designatas esse $M\Pi$ et $N\Pi$, ita ut sit $P = M\Pi$ et $Q = N\Pi$. Quare cum integrabilitas postulet hanc aequationem:

$$\partial P + 2Q\,\partial p + p\,\partial Q = 0,$$

facta hac substitutione nanciscemur sequentem aequationem:

$$M\partial\Pi + \Pi\partial M + 2N\Pi\,\partial p + Np\,\partial\Pi + \Pi p\,\partial N = 0,$$

ex qua, quia M et N sunt functiones cognitae ipsius p, elicimus

$$\frac{\partial\Pi}{\Pi} = \frac{-\partial M - 2N\,\partial p - p\,\partial N}{M + Np},$$

unde colligimus integrando

$$l\Pi = -l(M + Np) - \int\frac{N\,\partial p}{M + Np}.$$

Ponamus igitur brevitatis gratia

$$\int\frac{N\,\partial p}{M + Np} = lK,$$

quandoquidem etiam haec formula K tanquam data spectari potest, sicque erit $l\Pi = -l(M + Np) - lK + lA$. Quocirca pro solutione nostri problematis habebimus:

$$\Pi = \frac{A}{K(M + Np)}, \quad \text{existente} \quad lK = \int\frac{N\,\partial p}{M + Np}.$$

20. Invento autem hoc valore functionis quaesitae

$$\Pi = \frac{A}{K(M + Np)},$$

quoniam supra integrale in genere prodiit

$$P(px - y) + Qpp\left(x - \frac{y}{p}\right) = (px - y)(P + Qp),$$

substitutis pro P et Q debitis valoribus integrale formulae differentialis propositae $(Mx+Ny)\Pi\partial p$ erit

$$(px-y)(M\Pi+N\Pi p)=\frac{A(px-y)(M+Np)}{K(M+Np)},$$

quae commode ulterius reducitur ad hanc formam simplicissimam:

$$\frac{A(px-y)}{K},$$

sicque erit

$$\int\frac{(Mx+Ny)\partial p}{K(M+Np)}=\frac{px-y}{K},$$

existente

$$lK=\int\frac{N\partial p}{M+Np}\quad\text{sive}\quad K=e^{\int\frac{N\partial p}{M+Np}},$$

id quod operae pretium erit exemplis illustrare.

EXEMPLUM 1

21. Sit $M=1$ et $N=1$, ita ut proponatur haec formula differentialis: $(x+y)\Pi\partial p$. Hic igitur erit

$$lK=\int\frac{\partial p}{1+p}=l(1+p),$$

ideoque $K=1+p$, ita ut iam functio quaesita sit $\Pi=\frac{A}{(1+p)^2}$, hincque formula differentialis integrationem admittens erit $\frac{(x+y)\partial p}{(1+p)^2}$, quippe cuius integrale est $\frac{px-y}{1+p}$. Quodsi enim haec formula differentietur, prodit

$$\frac{x\partial p}{1+p}-\frac{(px-y)\partial p}{(1+p)^2},$$

quae reducitur ad hanc formam:

$$\frac{(x+y)\partial p}{(1+p)^2}.$$

EXEMPLUM 2

22. Sint ambae functiones M et N constantes, scilicet $M = m$ et $N = n$, ut proposita sit haec formula differentialis: $(mx + ny)\Pi \partial p$. Hic igitur erit primo

$$lK = \int \frac{n\partial p}{m + np} = l(m + np),$$

ita ut sit $K = m + np$. Hinc igitur functio quaesita Π erit

$$\frac{A}{(m + np)^2},$$

ita ut iam integrabilis sit haec formula:

$$\frac{(mx + ny)\partial p}{(m + np)^2},$$

quippe cuius integrale erit

$$\frac{px - y}{m + np}.$$

EXEMPLUM 3

23. Sumamus nunc $M = 1$ et $N = p$, ut formula integrabilis reddenda sit $(x + py)\Pi \partial p$. Hic igitur erit primo

$$lK = \int \frac{p\partial p}{1 + pp} = l\sqrt{(1 + pp)},$$

ideoque $K = \sqrt{(1 + pp)}$, unde fit functio quaesita

$$\Pi = \frac{A}{(1 + pp)^{\frac{3}{2}}},$$

hincque formula differentialis integrationem admittens erit

$$\frac{(x + py)\partial p}{(1 + pp)^{\frac{3}{2}}},$$

quae est ea ipsa, quam initio sumus contemplati, cuius ergo integrale est

$$\frac{px - y}{\sqrt{(1 + p^2)}}.$$

EXEMPLUM 4

24. Sit nunc $M = m$ et $N = np$, ut formula integrabilis reddenda sit $(mx + npy)\Pi\partial p$. Hic igitur erit

$$lK = \int \frac{np\partial p}{m + npp} = l\sqrt{(m + npp)},$$

ideoque $K = \sqrt{(m + npp)}$, unde functio quaesita erit

$$\Pi = \frac{A}{(m + npp)^{\frac{3}{2}}},$$

ita ut iam integrabilis sit haec formula

$$\frac{(mx + npy)\partial p}{(m + npp)^{\frac{3}{2}}},$$

cuius ergo integrale erit

$$\frac{px - y}{\sqrt{(m + npp)}}.$$

EXEMPLUM 5

25. Sit nunc $M = m$ et $N = np^{\lambda-1}$, ita ut formula integrabilis reddenda sit $(mx + np^{\lambda-1}y)\Pi\partial p$. Hic igitur erit

$$lK = \int \frac{np^{\lambda-1}\partial p}{m + np^{\lambda}} = \frac{1}{\lambda}l(m + np^{\lambda}),$$

ideoque $K = (m + np^{\lambda})^{\frac{1}{\lambda}}$, unde functio quaesita Π erit

$$\frac{A}{(m + np^{\lambda})^{\frac{\lambda+1}{\lambda}}},$$

ita ut iam integrabilis sit haec formula:

$$\frac{(mx + np^{\lambda-1}y)\partial p}{(m + np^{\lambda})^{\frac{\lambda+1}{\lambda}}},$$

erit

$$\frac{px - y}{(m + np^{\lambda})^{\frac{1}{\lambda}}}.$$

EXEMPLUM 6

26. Sit nunc $M = mp$ et $N = n$, ita ut formula integrabilis reddenda sit $(mpx + ny)\Pi\partial p$. Hic igitur erit

$$lK = \int \frac{n\partial p}{mp + np} = \frac{n}{m + n} lp,$$

ideoque $K = p^{\frac{n}{m+n}}$, ergo

$$\Pi = \frac{A}{(m + n)p^{\frac{m+2n}{m+n}}},$$

sicque formula integrabilis nunc est

$$\frac{(mpx + ny)\partial p}{(m + n)p^{\frac{m+2n}{m+n}}},$$

cuius ergo integrale erit

$$\frac{px - y}{p^{\frac{n}{m+n}}}.$$

27. Hic casus imprimis notabilis ocurrit, quo $m = -n$, sive $m + n = 0$; tum enim formula maxime incongrua resultat, ob exponentem ipsius p infinitum. Hic autem casus per se est obvius. Si enim quaeratur Π, ut ista formula $(px - y)\Pi\partial p$ evadat integrabilis, quoniam est $\partial\cdot(px - y) = x\partial p$, evidens est nullam dari functionem ipsius p tantum, qua huic conditioni satisfieri queat. Statim autem ac non fuerit $m + n = 0$, solutio semper est possibilis.

EXEMPLUM 7

28. Sumatur nunc $M = mpp$ et $N = n$, ut integrabilis reddi debeat haec formula: $(mppx + ny)\Pi\partial p$. Hic ergo erit

$$lK = \int \frac{n\partial p}{np + mpp} = lp - l(mp + n),$$

consequenter

$$K = \frac{p}{mp + n}, \quad \text{hincque} \quad \Pi = \frac{A}{pp},$$

sicque formula integrabilis iam erit

$$\frac{(mppx+ny)\partial p}{pp};$$

eius enim integrale erit

$$\frac{(px-y)(mp+n)}{p}.$$

EXEMPLUM 8

29. Sit nunc $M=p^{\lambda+1}$ et $N=1$, ita ut formula integrabilis reddenda sit $(p^{\lambda+1}x+y)\Pi\partial p$. Hic ergo erit

$$lK=\int\frac{\partial p}{p^{\lambda+1}+p}=lp-\frac{1}{\lambda}l(p^\lambda+1),$$

ergo

$$K=\frac{p}{(p^\lambda+1)^{\frac{1}{\lambda}}},$$

hincque

$$\Pi=\frac{A(p^\lambda+1)^{\frac{1-\lambda}{\lambda}}}{pp},$$

unde formula integrabilis erit

$$\frac{(p^\lambda+1)^{\frac{1-\lambda}{\lambda}}(p^{\lambda+1}x+y)\partial p}{pp},$$

quippe cuius integrale est

$$\frac{(px-y)(p^\lambda+1)^{\frac{1}{\lambda}}}{p}.$$

EXEMPLUM 9

30. Sit denique $M=mp^{\lambda+1}$ et $N=n$, ut formula integrabilis reddenda sit $(mp^{\lambda+1}x+ny)\Pi\partial p$. Hic ergo erit

$$lK=\int\frac{n\,\partial p}{mp^{\lambda+1}+np}=lp-\frac{1}{\lambda}l(mp^\lambda+n),$$

ideoque

$$K=\frac{p}{(mp^\lambda+n)^{\frac{1}{\lambda}}},$$

hincque

$$\Pi = \frac{A(mp^{\lambda}+n)^{\frac{1-\lambda}{\lambda}}}{pp},$$

unde formula integrabilis erit

$$\frac{(mp^{\lambda+1}x+ny)(mp^{\lambda}+n)^{\frac{1-\lambda}{\lambda}}\partial p}{pp},$$

quippe cuius integrale erit

$$\frac{(px-y)(mp^{\lambda}+n)^{\frac{1}{\lambda}}}{p}.$$

PROBLEMA 3

Invenire duas functiones ipsius p, quae sint P et Q, ut ista formula differentialis: $(px-y)^{n-1}(Px+Qy)\partial p$ fiat integrabilis.

SOLUTIO

31. Cum sit $x\partial p = \partial\cdot(px-y)$, erit

$$\int Px\partial p(px-y)^{n-1} = \frac{1}{n}P(px-y)^{n} - \frac{1}{n}\int(px-y)^{n}\partial P.$$

Deinde cum sit

$$\frac{y\partial p}{pp} = \partial\cdot\left(x-\frac{y}{p}\right),$$

loco $Qy\partial p$ scribamus $Qpp\cdot\frac{y\partial p}{pp}$, tum vero loco $px-y$ scribamus $p\left(x-\frac{y}{p}\right)$, ideoque loco $(px-y)^{n-1}$ scribendum erit $p^{n-1}\left(x-\frac{p}{y}\right)^{n-1}$. Hinc ergo pro altera parte habebimus

$$Qy\partial p(px-y)^{n-1} = Qpp\cdot\frac{y\partial p}{pp}\cdot p^{n-1}\left(x-\frac{y}{p}\right)^{n-1} = Qp^{n+1}\cdot\frac{y\partial p}{pp}\cdot\left(x-\frac{y}{p}\right)^{n-1},$$

hincque per reductionem erit

$$\int Qy\partial p(px-y)^{n-1} = \frac{1}{n}Qp^{n+1}\left(x-\frac{y}{p}\right)^{n} - \frac{1}{n}\int\left(x-\frac{y}{p}\right)^{n}\partial\cdot Qp^{n+1}.$$

32. Nunc igitur ut formula proposita integrationem admittat, necesse est, ut binae partes posteriores summatoriae ad nihilum redigantur, unde oritur ista aequatio:

$$(px-y)^n \partial P + \left(x - \frac{y}{p}\right)^n \partial \cdot Q p^{n+1} = 0,$$

hincque dividendo per $(px-y)^n$ erit

$$p^n \partial P + \partial \cdot Q p^{n+1} = 0,$$

cuius evolutio praebet

$$\partial P + p \partial Q + (n+1) Q \partial p = 0,$$

qua aequatione relatio requisita inter P et Q continetur; unde ergo data altera simul altera determinari potest; tum autem ipsum integrale formulae propositae erit

$$\frac{1}{n} P (px-y)^n + \frac{1}{n} Q p^{n+1} \left(x - \frac{y}{p}\right)^n,$$

sive

$$\frac{1}{n} (px-y)^n (P + Qp).$$

PROBLEMA 4

Si M et N designent functiones quascunque datas ipsius p, invenire eiusdem quantitatis functionem Π, ut ista formula differentialis: $(px-y)^{n-1}(Mx+Ny)\Pi \partial p$ fiat integrabilis.

SOLUTIO

33. Solutio praecedentis problematis huc transferetur statuendo $P = M\Pi$ et $Q = N\Pi$, unde conditio ante inventa ad hanc aequationem perducet:

$$M \partial \Pi + \Pi \partial M + N p \partial \Pi + \Pi p \partial N + (n+1) N \Pi \partial p = 0,$$

ex qua reperitur

$$\frac{\partial \Pi}{\Pi} = \frac{-\partial M - p \partial N - (n+1) N \partial p}{M + Np},$$

$$'\Pi = -l(M + Np) - n \int \frac{N \partial p}{M + Np}.$$

34. Ponamus iam, ut supra fecimus

$$\int \frac{N\partial p}{M+Np} = lK,$$

atque ad numeros procedendo erit

$$\Pi = \frac{A}{K^n(M+Np)},$$

sicque formula nostra integrabilis erit

$$\frac{(px-y)^{n-1}(Mx+Ny)\partial p}{K^n(M+Np)}.$$

Eius enim integrale erit

$$\frac{1}{n}\frac{(px-y)^n(M+Np)}{K^n(M+Np)} = \frac{(px-y)^n}{nK^n},$$

unde sumto $n = 1$ manifesto casus problematis tertii exsurgit.

35. Casus hic imprimis notatu dignus occurrit, quo $n = 0$; tum enim, ob $K^n = 1$, formula integrabilis reddita erit

$$\frac{(Mx+Ny)\partial p}{(M+Np)(px-y)}.$$

Eius vero integrale hinc videtur fieri infinitum, cuiusmodi valores ad logarithmos revocantur; formula enim $\frac{(px-y)^0}{0}$ [1]) aequivalet $l(px-y)$. Interim tamen hoc integrale neutiquam satisfacit, cuius rei ratio in evanescentia numeri n latet; reperitur autem haec formula differentialis resolvi in

$$\frac{x\partial p}{px-y} - \frac{N\partial p}{M+Np},$$

1) Non est rectum ponere $K^n = 1$ et considerare $(px-y)^n : n$, numero n evanescente. Oportet considerare

$$\frac{\left(\frac{px-y}{K}\right)^n - 1}{n}.$$

Quae expressio, n evanescente, aequivalet $l(px-y) - lK$. H. D.

unde si, ut fecimus, ponatur

$$\int \frac{N\partial p}{M+Np} = lK,$$

eius integrale erit $l(px-y)-lK$, ita ut hoc casu integrale sit $l\frac{px-y}{K}$. Reliquis autem casibus integralia erunt algebraica, cuius rei sequentia exempla perpendamus.

EXEMPLUM 1

36. Sit $M=1$ et $N=1$, eritque ut ante $lK = \int \frac{\partial p}{1+p} = l(1+p)$, ideoque $K = 1+p$, hincque $\Pi = \frac{A}{(1+p)^{n+1}}$, unde formula nostra integrabilis iam erit

$$\frac{(px-y)^{n-1}(x+y)\partial p}{(1+p)^{n+1}},$$

cuius integrale est

$$\frac{(px-y)^n}{n(1+p)^n}.$$

EXEMPLUM 2

37. Ponamus nunc $M=\alpha$ et $N=\beta$, ut formula integrabilis reddenda sit $(px-y)^{n-1}(\alpha x+\beta y)\Pi\partial p$. Hic ergo erit

$$lK = \int \frac{\beta\partial p}{\alpha+\beta p} = l(\alpha+\beta p),$$

ideoque $K=\alpha+\beta p$, hincque $\Pi = \frac{A}{(\alpha+\beta p)^{n+1}}$, unde nostra formula integrabilis reddenda erit

$$\frac{(px-y)^{n-1}(\alpha x+\beta y)\partial p}{(\alpha+\beta p)^{n+1}},$$

quippe cuius integrale est

$$\frac{(px-y)^n}{n(\alpha+\beta p)^n}.$$

EXEMPLUM 3

38. Sit nunc $M=1$ et $N=p$, ut formula integrabilis reddenda sit $(px-y)^{n-1}(x+py)\Pi\partial p$. Hic ergo erit

$$lK = \int \frac{p\partial p}{1+pp} = l\sqrt{(1+pp)},$$

ideoque $K = \sqrt{(1+pp)}$, hincque

$$\Pi = \frac{A}{(1+pp)^{\frac{n+2}{2}}},$$

sicque formula nostra integrabilis erit

$$\frac{(px-y)^{n-1}(x+py)\partial p}{(1+pp)^{\frac{n+2}{2}}},$$

eius enim integrale erit

$$\frac{(px-y)^n}{n(1+pp)^{\frac{n}{2}}}.$$

EXEMPLUM 4

39. Sit nunc $M = \alpha$ et $N = \beta p$, ut formula integrabilis reddenda sit $(px-y)^{n-1}(\alpha x + \beta py)\Pi\partial p$. Hic igitur erit

$$lK = \int \frac{\beta p\partial p}{\alpha + \beta pp} = \tfrac{1}{2}l(\alpha + \beta pp),$$

ideoque $K = \sqrt{(\alpha + \beta pp)}$, unde functio quaesita Π erit

$$\frac{A}{(\alpha + \beta pp)^{\frac{n+2}{2}}}.$$

Hinc formula nostra integrabilis erit

$$\frac{(px-y)^{n-1}(\alpha x + \beta py)\partial p}{(\alpha + \beta pp)^{\frac{n+2}{2}}},$$

quippe cuius integrale erit

$$\frac{(px-y)^n}{n(\alpha + \beta pp)^{\frac{n}{2}}}.$$

EXEMPLUM 5

40. Sit $M = \alpha$ et $N = \beta p^{\lambda-1}$, ut formula integrabilis reddenda sit $(px - y)^{n-1}(\alpha x + \beta p^{\lambda-1} y)\Pi \partial p$. Hic ergo erit

$$lK = \int \frac{\beta p^{\lambda-1}\partial p}{\alpha + \beta p^{\lambda}} = \frac{1}{\lambda} l(\alpha + \beta p^{\lambda}),$$

ideoque $K = (\alpha + \beta p^{\lambda})^{\frac{1}{\lambda}}$, unde functio quaesita Π erit

$$\frac{A}{(\alpha + \beta p^{\lambda})^{\frac{n+\lambda}{\lambda}}},$$

sicque formula nostra integrabilis erit

$$\frac{(px - y)^{n-1}(\alpha x + \beta p^{\lambda-1} y)\partial p}{(\alpha + \beta p^{\lambda})^{\frac{n+\lambda}{\lambda}}},$$

quippe cuius integrale est

$$\frac{(px - y)^{n}}{n(\alpha + \beta p^{\lambda})^{\frac{n}{\lambda}}}.$$

EXEMPLUM 6

41. Sit nunc $M = \alpha p$ et $N = \beta$, ita ut formula integrabilis reddenda sit $(px - y)^{n-1}(\alpha p x + \beta y)\Pi \partial p$. Hic igitur erit

$$lK = \int \frac{\beta \partial p}{\alpha p + \beta p} = \frac{\beta}{\alpha + \beta} lp,$$

ideoque $K = p^{\frac{\beta}{\alpha+\beta}}$. Hinc igitur functio proposita Π erit

$$\Pi = \frac{A}{(\alpha + \beta) p^{\frac{\alpha + (n+1)\beta}{\alpha+\beta}}},$$

sicque formula integrabilis nunc erit

$$\frac{(px - y)^{n-1}(\alpha p x + \beta y)\partial p}{(\alpha + \beta) p^{\frac{\alpha + (n+1)\beta}{\alpha+\beta}}},$$

cuius ergo integrale est

$$\frac{(px-y)^n}{np^{\frac{\beta n}{\alpha+\beta}}}.$$

EXEMPLUM 7

42. Sumatur nunc $M = \alpha pp$ et $N = \beta$, ut integrabilis reddi debeat haec formula: $(px-y)^{n-1}(\alpha ppx + \beta y)\Pi\partial p$. Hic ergo erit

$$lK = \int \frac{\beta\partial p}{\alpha pp + \beta p} = lp - l(\alpha p + \beta),$$

consequenter $K = \frac{p}{\alpha p + \beta}$, hincque

$$\Pi = \frac{A(\alpha p + \beta)^{n-1}}{p^{n+1}},$$

sicque formula integrabilis iam erit

$$\frac{(px-y)^{n-1}(\alpha ppx + \beta y)(\alpha p + \beta)^{n-1}\partial p}{p^{n+1}},$$

quippe cuius integrale est

$$\frac{(px-y)^n(\alpha p + \beta)^n}{np^n}.$$

EXEMPLUM 8

43. Sit nunc $M = p^{\lambda+1}$ et $N = 1$, ita ut formula integrabilis reddenda sit $(px-y)^{n-1}(p^{\lambda+1}x + y)\Pi\partial p$. Hic ergo erit

$$lK = \int \frac{\partial p}{p^{\lambda+1} + p} = lp - \frac{1}{\lambda}l(p^\lambda + 1),$$

consequenter $K = \frac{p}{(p^\lambda + 1)^{\frac{1}{\lambda}}}$, hincque

$$\Pi = \frac{A(p^\lambda + 1)^{\frac{n-\lambda}{\lambda}}}{p^{n+1}},$$

unde formula integrabilis erit

$$\frac{(px-y)^{n-1}(p^{\lambda+1}x+y)(p^{\lambda}+1)^{\frac{n-\lambda}{\lambda}}\partial p}{p^{n+1}},$$

quippe cuius integrale erit

$$\frac{(px-y)^{n}(p^{\lambda}+1)^{\frac{n}{\lambda}}}{np^{n}}.$$

EXEMPLUM 9

44. Sit denique $M=\alpha p^{\lambda+1}$ et $N=\beta$, ut formula integrabilis reddenda sit $(px-y)^{n-1}(\alpha p^{\lambda+1}x+\beta y)\Pi\partial p$. Hic ergo erit

$$lK=\int\frac{\beta\partial p}{\alpha p^{\lambda+1}+\beta p}=lp-\frac{1}{\lambda}l(\alpha p^{\lambda}+\beta),$$

ideoque $K=\dfrac{p}{(\alpha p^{\lambda}+\beta)^{\frac{1}{\lambda}}}$, hincque

$$\Pi=\frac{A(\alpha p^{\lambda}+\beta)^{\frac{n-\lambda}{\lambda}}}{p^{n+1}},$$

unde formula integrabilis erit

$$\frac{(px-y)^{n-1}(\alpha p^{\lambda+1}x+\beta y)(\alpha p^{\lambda}+\beta)^{\frac{n-\lambda}{\lambda}}\partial p}{p^{n+1}},$$

cuius ergo integrale erit

$$\frac{(px-y)^{n}(\alpha p^{\lambda}+\beta)^{\frac{n}{\lambda}}}{np^{n}}.$$

OBSERVATIO SINGULARIS CIRCA AEQUATIONES DIFFERENTIALES LINEARES

Conventui exhibita die 19. Martii 1778
Commentatio 720 indicis ENESTROEMIANI
Nova acta academiae scientiarum Petropolitanae **14** (1797/8), 1805, p. 52—61
Summarium ibidem p. 65

SUMMARIUM

Soit une équation différentielle linéaire, ou du premier degré, à résoudre:

$$pz + qdz + rddz + sd^3z + \text{ etc.} = 0,$$

où p, q, r etc. sont fonctions d'une autre variable x, dont la différentielle est constante, et soit m le multiplicateur, fonction de x, qui rend cette équation intégrable, on parvient à une équation

$$Pm + Qdm + Rddm + Sd^3m + \text{etc.} = 0$$

que l'auteur appelle la *resolvante* de l'équation proposée $pz + qdz + rddz + sd^3z + \text{etc.} = 0$; et ces deux équations, que feu M. EULER appelle *conjuguées*, sont entr'elles dans une telle liaison réciproque, que si l'une est résoluble, l'autre le sera nécessairement aussi; et c'est là l'observation qui a fourni le sujet de ce mémoire.

1. Hoc nomine hodie designari solent aequationes differentiales huius formae:

$$pz + q\frac{dz}{dx} + r\frac{ddz}{dx^2} + s\frac{d^3z}{dx^3} + t\frac{d^4z}{dx^4} + \text{etc.} = 0,$$

ubi variabilis z in singulis terminis unicam tantum tenet dimensionem, litterae vero p, q, r, s etc. denotant functiones quascunque alterius variabilis x, cuius

differentiale constans assumitur, quod, quia in singulis terminis homogeneitatem complere debet, brevitatis gratia ubique omittere licebit, sicque illam aequationem in posterum hac forma repraesentabimus:

$$pz + qdz + rddz + sd^3z + td^4z + \text{etc.} = 0.$$

2. Constat autem huiusmodi aequationes duobus casibus in genere resolvi posse, altero, quo singulae litterae p, q, r, s, t etc. sunt quantitates constantes, altero vero, quo aequationem homogeneam reddunt; priore ergo casu aequatio hanc habebit formam:

$$\alpha z + \beta dz + \gamma ddz + \delta d^3z + \varepsilon d^4z + \text{etc.} = 0,$$

cui semper huiusmodi forma satisfacit $z = e^{\lambda x}$, quippe quo valore substituto, factaque divisione per $e^{\lambda x}$, oritur aequatio mere algebraica, ista scilicet:

$$\alpha + \beta\lambda + \gamma\lambda^2 + \delta\lambda^3 + \varepsilon\lambda^4 + \zeta\lambda^5 + \text{etc.} = 0,$$

ex qua ergo tot valores pro littera λ erui poterunt, quoti ordinis fuerit aequatio differentialis, qui valores si fuerint a, b, c, d, e etc., hinc adeo integrale completum exhiberi poterit, ita expressum:

$$z = Ae^{ax} + Be^{bx} + Ce^{cx} + De^{dx} + \text{etc.},$$

ubi litterae A, B, C, D etc. sunt constantes arbitrariae per integrationes ingressae.

3. Pro altero casu, quo aequatio evadit homogenea, eam semper ad hanc formam reducere licet:

$$\alpha z + \beta x dz + \gamma xx ddz + \delta x^3 d^3z + \varepsilon x^4 d^4z + \text{ etc.} = 0,$$

cui semper satisfacit huiusmodi forma $z = x^\lambda$. Hinc iterum oritur aequatio algebraica ista, postquam scilicet per x^λ fuerit divisa:

$$\begin{gathered}\alpha + \beta\lambda + \gamma\lambda(\lambda - 1) + \delta\lambda(\lambda - 1)(\lambda - 2) + \varepsilon\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3) \\ + \zeta\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4) + \text{etc.} = 0,\end{gathered}$$

unde iterum tot valores pro λ eruuntur, quoti gradus fuerit aequatio, qui valores si fuerint a, b, c, d, e etc., habebitur integrale completum

$$z = Ax^a + Bx^b + Cx^c + Dx^d + Ex^e + \text{etc.}$$

4. Praeter hos duos casus raro tales aequationes occurrere solent, quas quidem resolvere seu integrare liceat, nisi forte data opera a posteriori fabricentur. Interim tamen ab acutissimis Geometris Gallis non ita pridem insignia subsidia circa tales aequationes ingeniosissime sunt excogitata; quae cum perlustrassem, in observationem quandam prorsus singularem incidi, quae mihi quidem in hoc genere maximi momenti videtur, et quam propterea hic in medium proferre constitui.

5. Proposita autem tali forma generali:

$$pz + qdz + rddz + sd^3z + td^4z = 0,$$

quae usque ad quartum gradum assurgat, quandoquidem deinceps calculum ad quosvis alios gradus accommodare facile erit, quaero eiusmodi functionem ipsius x, in quam si ista forma ducatur, integrabilis evadat. Sit igitur iste multiplicator $= m$, ita ut integrabilis esse debeat haec forma:

$$mpz + mqdz + mrddz + msd^3z + mtd^4z = 0,$$

atque duplici modo ad integrale perveniri potest, prouti vel a primo termino vel ab ultimo integratio inchoetur; ubi quidem sponte intelligitur, insuper per elementum dx multiplicationem esse instituendam, quod autem brevitatis gratia omittimus.

6. Incipiamus primo a termino ultimo mtd^4z, qui pro integrali praebet terminum mtd^3z; huius ergo differentiale auferatur et remanebit haec forma:

$$mpz + mqdz + mrddz + d^3z(ms - d \cdot mt).$$

Iam ex hoc postremo membro in integrale ingredietur

$$ddz(ms - d \cdot mt);$$

huius iam differentiale iterum auferatur, et remanebit

$$mpz + mqdz + ddz(mr - d \cdot ms + dd \cdot mt).$$

Hinc iam novus terminus in integrale ingrediens erit

$$dz(mr - d \cdot ms + dd \cdot mt),$$

cuius differentiale allatum relinquit adhuc

$$mpz + dz(mq - d \cdot mr + dd \cdot ms - d^3 \cdot mt),$$

unde ultimum integralis membrum erit

$$z(mq - d \cdot mr + dd \cdot ms - d^3 \cdot mt),$$

cuius ergo differentiale si auferatur, nihil relinqui debebit, unde resultabit ista aequatio:

$$mp - d \cdot mq + dd \cdot mr - d^3 \cdot ms + d^4 \cdot mt = 0.$$

7. Hoc igitur modo deducti sumus ad aequationem differentialem pariter quarti ordinis, ex qua valorem multiplicatoris quaesiti m investigari oportet, quippe quo invento forma integrata erit

$$mtd^3z + ddz(ms - d \cdot mt) + dz(mr - d \cdot ms + dd \cdot mt) + z(mq - d \cdot mr + dd \cdot ms - d^3 \cdot mt),$$

aequatio autem, unde valorem ipsius m erui convenit, est, uti invenimus,

$$mp - d \cdot mq + dd \cdot mr - d^3 \cdot ms + d^4 \cdot mt = 0.$$

8. At vero inverso ordine integrationem a termino primo inchoare licet, siquidem hinc orietur prima integralis pars $z\int mp$, cuius differentiale subtractum relinquet

$$dz(mq - \int mp) + mrddz + msd^3z + mtd^4z.$$

Ex prima parte deducitur secunda integralis pars

$$dz(\int mq - \int\int mp).$$

Iam huius differentiale ablatum relinquet

$$ddz(mr - \int mq + \int\int mp) + msd^3z + mtd^4z.$$

Hinc igitur nascitur tertia integralis pars

$$ddz(\int mr - \int\int mq + \int\int\int mp),$$

huius porro differentiale ablatum relinquet

$$d^3z(ms - \int mr + \int\int mq - \int^3 mp) + mtd^4z\,,$$

unde postrema integralis pars colligitur:

$$d^3z(\int ms - \int\int mr + \int^3 mq - \int^4 mp),$$

quocirca si huius differentiale auferatur, nihil relinqui debet, sicque pervenietur ad hanc aequationem:

$$mt - \int ms + \int\int mr - \int^3 mq + \int^4 mp = 0,$$

unde pariter valorem multiplicatoris quaesiti m investigari oportet.

9. Evidens autem est hanc aequationem posteriorem egregie cum priore consentire, nam haec aequatio differentiata praebet

$$d \cdot mt - ms + \int mr - \int\int mq + \int^3 mp = 0,$$

haec autem porro differentiata praebet

$$dd \cdot mt - d \cdot ms + mr - \int mq + \int\int mp = 0,$$

quae rursus differentiata dat

$$d^3 \cdot mt - d^2 \cdot ms + d \cdot mr + mq - \int mp = 0,$$

cuius denique differentiale producit ipsam aequationem prius inventam:

$$d^4 \cdot mt - d^3 \cdot ms + dd \cdot mr - d \cdot mq + mp = 0.$$

10. Evolvamus nunc hanc aequationem, ex qua multiplicatorem quaesitum m erui oportet, unde nascetur sequens aequatio:

$$\left.\begin{array}{llll} mp - qdm & + rddm & - sd^3m + td^4m \\ - mdq + 2drdm & - 3dsddm + 4dtd^3m & & \\ + mddr - 3dmdds & + 6ddmddt & & \\ - md^3s + 4dmd^3t & & & \\ + md^4t & & & \end{array}\right\} = 0.$$

11. Quodsi ergo hanc postremam aequationem resolvere, vel saltem quodpiam integrale particulare eruere licuerit, talis valor ipsius m, si in aequationem primo propositam multiplicetur, eam integrabilem reddet, ita ut hoc modo per unam integrationem ad gradum inferiorem differentialitatis reducatur. Hanc ob rationem aequationem ultimo inventam vocabimus *aequationem resolventem* formae propositae

$$pz + qdz + rddz + sd^3z + td^4z = 0,$$

atque hinc patet, quomodo pro quavis aequatione differentiali lineari proposita eius resolventem inveniri oporteat.

12. Quo autem formam aequationis resolventis clarius perspiciamus, eam hac forma repraesentemus:

$$Pm + Qdm + Rddm + Sd^3m + Td^4m = 0,$$

ut ad similitudinem aequationis propositae revocetur; atque comparatione instituta quantitates P, Q, R, S et T per cognitas, quae sunt p, q, r, s et t, sequenti modo determinabuntur:

$$\begin{aligned} P &= p - dq + ddr - d^3s + d^4t \\ Q &= - q + 2dr - 3dds + 4d^3t \\ R &= r - 3ds + 6ddt \\ S &= - s + 4dt \\ T &= t. \end{aligned}$$

Cuius relationis ope pro quovis casu aequatio resolvens nullo labore exhiberi poterit.

13. Quodsi aequationem resolventem tanquam datam spectare velimus, ex ea vicissim ipsa aequatio prior, cuius est resolvens, exhiberi poterit per sequentes relationes, quibus litterae minusculae p, q, r etc. per maiusculas P, Q, R etc. determinantur:

$$\begin{aligned} t &= T \\ s &= -S + 4dT \\ r &= R - 3dS + 6ddT \\ q &= -Q + 2dR - 3ddS + 4d^3T \\ p &= P - dQ + ddR - d^3S + d^4T. \end{aligned}$$

Unde patet litteras minusculas eadem prorsus lege a maiusculis pendere, qua hae ab illis pendere sunt inventae.

14. Quaelibet igitur huiusmodi aequatio differentialis cum sua resolvente tali reciproco nexu coniugatur, ut si una fuerit resolvens alterius, vicissim quoque haec resolvens sit illius. Cum igitur huiusmodi binae aequationes tam insigni vinculo sint inter se connexae, eas inter se *coniugatas* appellemus, ita ut quaelibet aequatio huius indolis suam habeat coniugatam, atque utraque ope alterius resolvi possit. Quodsi enim alterutra resolutionem admittat, simul quoque alterius resolutio semper est in potestate, atque in hoc consistit observatio illa singularis, quae mihi quidem maxime notatu digna videtur; neque enim memini eam a quoquam alio factam vidisse.

15. Scribamus nunc uniformitatis gratia litteram Z loco m, atque binae huiusmodi aequationes coniugatae, cuiuscunque fuerint ordinis, sequenti modo repraesententur:

$$\begin{aligned} &pz + qdz + rddz + sd^3z + td^4z + vd^5z + \text{etc.} = 0 \\ &PZ + QdZ + Rd^2Z + Sd^3Z + Td^4Z + Vd^5Z + \text{etc.} = 0 \end{aligned}$$

ac primo quidem litterae maiusculae sequenti lege per minusculas determinabuntur:

$$\begin{aligned} P &= p - dq + ddr - d^3s + d^4t - d^5v + d^6w \text{ etc.} \\ Q &= -q + 2dr - 3dds + 4d^3t - 5d^4v + 6d^5w \text{ etc.} \\ R &= r - 3ds + 6ddt - 10d^3v + 15d^4w \text{ etc.} \\ S &= -s + 4dt - 10ddv + 20d^3w \text{ etc.} \\ T &= t - 5dv + 15ddw \text{ etc.} \\ V &= -v + 6dw \text{ etc.} \\ W &= w \text{ etc.} \end{aligned}$$

Prorsus autem secundum eandem legem litterae minusculae per maiusculas determinabuntur; erit enim:

$$\begin{aligned}
p &= P - dQ + ddR - d^3S + d^4T - d^5V + d^6W \text{ etc.}\\
q &= -Q + 2dR - 3ddS + 4d^3T - 5d^4V + 6d^5W \text{ etc.}\\
r &= R - 3dS + 6ddT - 10d^3V + 15d^4W \text{ etc.}\\
s &= -S + 4dT - 10ddV + 20d^3W \text{ etc.}\\
t &= T - 5dV + 15ddW \text{ etc.}\\
v &= -V + 6dW \text{ etc.}\\
w &= W \text{ etc.}
\end{aligned}$$

16. Applicemus hoc ad binos casus initio memoratos, quorum resolutio nulla laborat difficultate, atque prioris aequationis

$$\alpha z + \beta dz + \gamma ddz + \delta d^3z + \varepsilon d^4z \text{ etc.} = 0$$

coniugata erit

$$\alpha Z - \beta dZ + \gamma ddZ - \delta d^3Z + \varepsilon d^4Z \text{ etc.} = 0,$$

quae ab illa non discrepat nisi alternatione signorum; pro alterius vero

$$\alpha z + \beta x dz + \gamma xx ddz + \delta x^3 d^3z + \varepsilon x^4 d^4z \text{ etc.} = 0$$

coniugata habebimus

$$\begin{aligned}
P &= \alpha - \beta + 2\gamma - 6\delta + 24\varepsilon \text{ etc.}\\
Q &= -x(\beta - 2 \cdot 2\gamma + 3 \cdot 6\delta - 4 \cdot 24\varepsilon \text{ etc.})\\
R &= xx(\gamma - 3 \cdot 3\delta + 6 \cdot 12\varepsilon \text{ etc.})\\
S &= -x^3(\delta - 4 \cdot 4\varepsilon \text{ etc.})\\
T &= x^4\ (\varepsilon \text{ etc.}).
\end{aligned}$$

Unde patet aequationem coniugatam etiam esse homogeneam.

17. Quodsi ergo binarum talium aequationum coniugatarum alterutra resolutionem admittat, tum altera quoque, saltem semel, integrari poterit; quo facto, si insuper huius aequationis integratae coniugata resolutionem admittat, denuo integratio succedet, atque ita porro, quae methodus utique foret non parum operosa. Verum hic imprimis observandum occurrit, quotiescunque integrale completum aequationis resolventis innotuerit, tum ope facilis operationis non solum ipsius aequationis propositae

$$pz + qdz + rddz + sd^3z + \text{etc.} = 0$$

integrale completum, sed etiam integrale huius aequationis multo generalioris:

$$pz + qdz + rddz + sd^3z + \text{etc.} = X$$

exhiberi posse, denotante X functionem quamcunque ipsius x, id scilicet integrale, quod absolutis tot integrationibus, quoti gradus fuerit differentialitas, tandem reperiretur, idque adeo ope unicae tantum integrationis, quod, cum plurimum in recessu habere possit, hic data opera explicabo.

18. Sufficiet autem aequationem differentialem tertii tantum gradus sumsisse, ita ut resolvenda proponatur haec aequatio:

$$p + qdz + rddz + sd^3z = X,$$

cui respondeat ista coniugata:

$$PZ + QdZ + RddZ + Sd^3Z = 0,$$

cuius integrale completum, quoniam tres constantes arbitrarias involvere debet, ita repraesentari poterit: $Z = A\Phi + B\Phi' + C\Phi''$, ubi ergo litterae Φ, Φ', Φ'' certae erunt functiones ipsius x, quae singulae idoneos multiplicatores pro aequatione proposita praebebunt: multiplicatione autem per Φ facta et integratione instituta ponamus resultare hanc aequationem:

$$\text{♄}\, z + \text{♃}\, dz + \text{♂}\, ddz = \int X\Phi dx\,.$$

Simili vero modo sumto Φ' pro multiplicatore prodeat haec aequatio:

$$\text{♄}'z + \text{♃}'dz + \text{♂}'ddz = \int X\Phi' dx\,.$$

Eodemque modo ex multiplicatore Φ'' prodeat

$$\text{♄}''z + \text{♃}''dz + \text{♂}''ddz = \int X\Phi'' dx\,,$$

ubi hae tres integrationes instar unicae spectari possunt.

19. Totum negotium ergo perduximus ad has tres aequationes:

$$\begin{array}{rl}
\text{I.} & \text{♄}z + \text{♃}\,dz + \text{♂}\,ddz = \int X\Phi dx\,, \\
\text{II.} & \text{♄}'z + \text{♃}'dz + \text{♂}'ddz = \int X\Phi' dx\,, \\
\text{III.} & \text{♄}''z + \text{♃}''dz + \text{♂}''ddz = \int X\Phi'' dx\,.
\end{array}$$

Ex his tribus aequationibus iam facile est binas formulas dz et ddz eliminare. Multiplicetur scilicet prima per $\odot$, secunda per $\odot'$, tertia per $\odot''$, quae quantitates ita capiantur, ut fiat

$$\odot\text{♃} + \odot'\text{♃}' + \odot''\text{♃}'' = 0,$$

et

$$\odot\text{♂} + \odot'\text{♂}' + \odot''\text{♂}'' = 0,$$

et hae aequationes in unam summam collectae dabunt

$$z(\odot\,\text{♄} + \odot'\,\text{♄}' + \odot''\,\text{♄}'') = \odot\int X\Phi dx + \odot'\int X\Phi' dx + \odot''\int X\Phi'' dx\,.$$

Sicque valor quantitatis z erit[1])

$$z = \frac{\odot\int X\Phi dx + \odot'\int X\Phi' dx + \odot''\int X\Phi'' dx}{\odot\,\text{♄} + \odot'\,\text{♄}' + \odot''\,\text{♄}''}$$

haecque expressio adeo est integrale completum aequationis differentialis propositae tertii ordinis, siquidem hic tria occurrunt signa summatoria, quorum quodvis involvit constantem arbitrariam.

1) Cf. p. 280. Vide quoque notam 2) p. 251. H. D.

RECHERCHES SUR QUELQUES INTEGRATIONS REMARQUABLES DANS L'ANALYSE DES FONCTIONS A DEUX VARIABLES CONNUES SOUS LE NOM DE DIFFERENCES PARTIELLES

Présenté à l'Académie le 8 Décembre 1777

Commentatio 724 indicis ENESTROEMIANI

Nova acta academiae scientiarum Petropolitanae **15** (1799/1802), 1806, p. 3—28

Résumé ibidem p. 93—94

RÉSUMÉ

Le but que l'auteur de ce mémoire avoit eu en vue, a été d'enseigner une méthode de trouver les intégrales complettes des équations différentielles suivantes:

$$P = x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 0,$$

$$Q = x^2\left(\frac{\partial\partial z}{\partial x^2}\right) + 2xy\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) + y^2\left(\frac{\partial\partial z}{\partial y^2}\right) = 0,$$

$$R = x^3\left(\frac{\partial^3 z}{\partial x^3}\right) + 3x^2y\left(\frac{\partial^3 z}{\partial x^2\partial y}\right) + 3xy^2\left(\frac{\partial^3 z}{\partial x\,\partial y^2}\right) + y^3\left(\frac{\partial^3 z}{\partial y^3}\right) = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$Z = x^\lambda\left(\frac{\partial^\lambda z}{\partial x^\lambda}\right) + \frac{\lambda}{1}\cdot x^{\lambda-1}\cdot y\left(\frac{\partial^\lambda z}{\partial x^{\lambda-1}\partial y}\right) + \frac{\lambda}{1}\cdot\frac{\lambda-1}{2}\cdot x^{\lambda-2}y^2\left(\frac{\partial^\lambda z}{\partial x^{\lambda-2}\partial y^2}\right) + \text{etc.} = 0.$$

Pour cet effet il commence par démontrer que chacune des expressions $Q, R, S, \ldots, Z$ peut être formée de celle qui la precède immédiatement, attendu que:

$$Q = x\left(\frac{\partial P}{\partial x}\right) + y\left(\frac{\partial P}{\partial y}\right) - 1 \cdot P,$$
$$R = x\left(\frac{\partial Q}{\partial x}\right) + y\left(\frac{\partial Q}{\partial y}\right) - 2 \cdot Q,$$
$$S = x\left(\frac{\partial R}{\partial x}\right) + y\left(\frac{\partial R}{\partial y}\right) - 3 \cdot R,$$
etc.

Et moyennant ces beaux rapports entre les quantités P, Q, R, etc. il est conduit à l'avantage de trouver les intégrales complettes des équations différentielles $P = 0$, $Q = 0$, $R = 0$, et ainsi de suite. Or la méthode dont feu M. EULER s'est servi dans ces intégrations exige pour chaque cas autant d'intégrations que le degré du différentiel indique, tandis que toutes ces solutions peuvent être exécutées plus facilement, moyennant une seule intégration, méthode qui a encore le grand avantage de s'étendre aussi à l'intégration des équations différentielles composées des précédentes et comprises sous la forme générale

$$Az + BP + CQ + DR + ES + \text{etc.}$$

Une suite de problèmes dans lesquels l'intégrale complette des équations de cette forme est cherchée, termine ce mémoire.

Prenant z pour marquer une fonction quelconque des deux variables x et y, on sait que la première différentiation, selon qu'on prend ou la seule x ou la seule y pour variable, fournit ces deux formules différentielles du premier degré: $\frac{\partial z}{\partial x}$ et $\frac{\partial z}{\partial y}$. La seconde différentiation donne ces trois formules différentielles du second ordre:

$$\frac{\partial\partial z}{\partial x^2}, \quad \frac{\partial\partial z}{\partial x\,\partial y}, \quad \frac{\partial\partial z}{\partial y^2}.$$

La troisième différentiation conduit à ces quatres formules différentielles du troisième degré:

$$\frac{\partial^3 z}{\partial x^3}, \quad \frac{\partial^3 z}{\partial x^2\partial y}, \quad \frac{\partial^3 z}{\partial x\,\partial y^2}, \quad \frac{\partial^3 z}{\partial y^3}.$$

La quatrième différentiation produit ces cinq différentielles du quatrième degré:

$$\frac{\partial^4 z}{\partial x^4}, \quad \frac{\partial^4 z}{\partial x^3\partial y}, \quad \frac{\partial^4 z}{\partial x^2\partial y^2}, \quad \frac{\partial^4 z}{\partial x\,\partial y^3}, \quad \frac{\partial^4 z}{\partial y^4},$$

et ainsi de suite. Nous omettons ici les guillemets, entre lesquels on a coûtume ordinairement de renfermer ces formules, puisque aucune ambiguité n'est à craindre dans les recherches que nous allons entreprendre.

Cela posé je considérerai ici les expressions suivantes[1]):

$$\text{I.}\quad P = x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y},$$

$$\text{II.}\quad Q = x^2\cdot\frac{\partial\partial z}{\partial x^2} + 2xy\cdot\frac{\partial\partial z}{\partial x\,\partial y} + y^2\cdot\frac{\partial\partial z}{\partial y^2},$$

$$\text{III.}\quad R = x^3\cdot\frac{\partial^3 z}{\partial x^3} + 3xxy\cdot\frac{\partial^3 z}{\partial x^2\partial y} + 3xyy\cdot\frac{\partial^3 z}{\partial x\,\partial y^2} + y^3\cdot\frac{\partial^3 z}{\partial y^3},$$

$$\text{IV.}\quad S = x^4\cdot\frac{\partial^4 z}{\partial x^4} + 4x^3y\cdot\frac{\partial^4 z}{\partial x^3\partial y} + 6xxyy\cdot\frac{\partial^4 z}{\partial x^2\partial y^2} + 4xy^3\cdot\frac{\partial^4 z}{\partial x\partial y^3} + y^4\cdot\frac{\partial^4 z}{\partial y^4},$$

et ainsi de suite. En général nous aurons celle-ci:

$$Z = x^\lambda\cdot\frac{\partial^\lambda z}{\partial x^\lambda} + \frac{\lambda}{1}\cdot x^{\lambda-1}y\cdot\frac{\partial^\lambda z}{\partial x^{\lambda-1}\partial y} + \frac{\lambda}{1}\cdot\frac{\lambda-1}{2}\cdot x^{\lambda-2}y^2\cdot\frac{\partial^\lambda z}{\partial x^{\lambda-2}\partial y^2} + \frac{\lambda}{1}\cdot\frac{\lambda-1}{2}\cdot\frac{\lambda-2}{3}\cdot x^{\lambda-3}y^3\cdot\frac{\partial^\lambda z}{\partial x^{\lambda-3}\partial y^3} + \text{etc.}$$

Ici j'observe d'abord, que chacune de ces expressions peut être formée de celle qui la précède immédiatement, et nous verrons qu'on aura toujours:

$$Q = x\cdot\frac{\partial P}{\partial x} + y\cdot\frac{\partial P}{\partial y} - 1P,$$

$$R = x\cdot\frac{\partial Q}{\partial x} + y\cdot\frac{\partial Q}{\partial y} - 2Q,$$

$$S = x\cdot\frac{\partial R}{\partial x} + y\cdot\frac{\partial R}{\partial y} - 3R,$$

$$T = x\cdot\frac{\partial S}{\partial x} + y\cdot\frac{\partial S}{\partial y} - 4S,$$

et ainsi de suite. Où il est à remarquer que si nous mettons O pour la formule qui précède la première P, nous aurons $O = z$, et partant

$$P = x\cdot\frac{\partial O}{\partial x} + y\cdot\frac{\partial O}{\partial y} - 0\cdot O.$$

Pour démontrer la vérité de toutes ces équations, commençons par la première, qui exprime la valeur de Q, et puisque $P = x\cdot\frac{\partial z}{\partial x} + y\cdot\frac{\partial z}{\partial y}$, la différentiation nous donnera

1) Comparer le mémoire 785 p. 443 de ce volume. H. D.

$$\frac{\partial P}{\partial x} = 1\cdot\frac{\partial z}{\partial x} + x\cdot\frac{\partial\partial z}{\partial x^2} + y\cdot\frac{\partial\partial z}{\partial x\partial y}$$

et

$$\frac{\partial P}{\partial y} = 1\cdot\frac{\partial z}{\partial y} + x\cdot\frac{\partial\partial z}{\partial x\partial y} + y\cdot\frac{\partial\partial z}{\partial y^2}.$$

De là nous tirerons cette équation:

$$x\cdot\frac{\partial P}{\partial x} + y\cdot\frac{\partial P}{\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + xx\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + yy\frac{\partial\partial z}{\partial y^2}$$

qui se reduit ouvertement à cette forme:

$$x\cdot\frac{\partial P}{\partial x} + y\cdot\frac{\partial P}{\partial y} = P + Q,$$

et partant on aura

$$Q = x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P.$$

Pour la seconde de nos équations, puisque nous avons supposé

$$Q = x^2\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + y^2\frac{\partial\partial z}{\partial x^2},$$

nous en tirons

$$\frac{\partial Q}{\partial x} = 2x\frac{\partial\partial z}{\partial x^2} + 2y\frac{\partial\partial z}{\partial x\partial y} + x^2\frac{\partial^3 z}{\partial x^3} + 2xy\frac{\partial^3 z}{\partial x^2\partial y} + y^2\frac{\partial^3 z}{\partial x\partial y^2},$$

$$\frac{\partial Q}{\partial y} = 2x\frac{\partial\partial z}{\partial x\partial y} + 2y\frac{\partial\partial z}{\partial y^2} + x^2\frac{\partial^3 z}{\partial x^2\partial y} + 2xy\frac{\partial^3 z}{\partial x\partial y^2} + y^2\frac{\partial^3 z}{\partial y^3}.$$

Maintenant la combinaison de ces formules fournira:

$$x\cdot\frac{\partial Q}{\partial x} + y\cdot\frac{\partial Q}{\partial y} = 2xx\cdot\frac{\partial\partial z}{\partial x^2} + 4xy\cdot\frac{\partial\partial z}{\partial x\partial y} + 2yy\cdot\frac{\partial\partial z}{\partial y^2}$$
$$+ x^3\cdot\frac{\partial^3 z}{\partial x^3} + 3xxy\cdot\frac{\partial^3 z}{\partial x^2\partial y} + 3xyy\cdot\frac{\partial^3 z}{\partial x\partial y^2} + y^3\cdot\frac{\partial^3 z}{\partial y^3}.$$

Cette équation se reduit évidemment à la suivante:

$$x\cdot\frac{\partial Q}{\partial x} + y\cdot\frac{\partial Q}{\partial y} = 2Q + R,$$

de sorte qu'il y a

$$R = x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} - 2Q.$$

Pour démontrer la vérité de la troisième de nos équations, puisque nous avons

$$R = x^3 \frac{\partial^3 z}{\partial x^3} + 3xxy \frac{\partial^3 z}{\partial x^2 \partial y} + 3xyy \frac{\partial^3 z}{\partial x \partial y^2} + y^3 \frac{\partial^3 z}{\partial y^3},$$

nous en tirons:

$$\frac{\partial R}{\partial x} = 3xx \frac{\partial^3 z}{\partial x^3} + 6xy \frac{\partial^3 z}{\partial x^2 \partial y} + 3yy \frac{\partial^3 z}{\partial x \partial y^2} + x^3 \frac{\partial^4 z}{\partial x^4} + 3xxy \frac{\partial^4 z}{\partial x^3 \partial y} + 3xyy \frac{\partial^4 z}{\partial x^2 \partial y^2} + y^3 \frac{\partial^4 z}{\partial x \partial y^3},$$

$$\frac{\partial R}{\partial y} = 3xx \frac{\partial^3 z}{\partial x^2 \partial y} + 6xy \frac{\partial^3 z}{\partial x \partial y^2} + 3yy \frac{\partial^3 z}{\partial y^3} + x^3 \frac{\partial^4 z}{\partial x^3 \partial y} + 3xxy \frac{\partial^4 z}{\partial x^2 \partial y^2} + 3xyy \frac{\partial^4 z}{\partial x \partial y^3} + y^3 \frac{\partial^4 z}{\partial x^4}.$$

Ces deux équations étant combinées, elles donnent:

$$\begin{aligned} x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} &= 3x^3 \frac{\partial^3 z}{\partial x^3} + 9xxy \frac{\partial^3 z}{\partial x^2 \partial y} + 9xyy \frac{\partial^3 z}{\partial x \partial y^2} + 3y^3 \frac{\partial^3 z}{\partial y^3} \\ &+ x^4 \frac{\partial^4 z}{\partial x^4} + 4x^3 y \frac{\partial^4 z}{\partial x^3 \partial y} + 6xxyy \frac{\partial^4 z}{\partial x^2 \partial y^2} + 4xy^3 \frac{\partial^4 z}{\partial x \partial y^3} + y^4 \frac{\partial^4 z}{\partial y^4}, \end{aligned}$$

équation qui se réduit encore évidemment à celle-ci:

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} = 3R + S,$$

d'où l'on tire par conséquent

$$S = x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} - 3R.$$

Il seroit superflu de démontrer par le même calcul la vérité des équations suivantes, puisqu'il est déjà assez clair, qu'on parviendra, par des opérations semblables, toujours à des équations telles que nous les avons assignées ci-dessus. Or ces beaux rapports entre les quantités P, Q, R etc. nous conduiront à l'avantage de trouver les intégrales, et même les intégrales complettes, des équations différentielles suivantes:

1°. $P = 0$, 2°. $Q = 0$, 3°. $R = 0$, 4°. $S = 0$, et ainsi de suite.

Pour cet effet nous n'avons qu'à résoudre les trois problèmes préliminaires suivans.

PROBLEME PRELIMINAIRE I

Trouver une fonction des deux variables x et y, qui soit v, telle qu'il devienne

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=0.$$

SOLUTION

Puisque v est fonction de x et y, supposons qu'en la différentiant, en prenant tant x que y variable, on trouve

$$\partial v = p\,\partial x + q\,\partial y,$$

de sorte que $p=\frac{\partial v}{\partial x}$ et $q=\frac{\partial v}{\partial y}$, et partant il faudra satisfaire à cette équation:

$$x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}=xp+yq=0,$$

d'où l'on tire $q=-\frac{xp}{y}$. Cette valeur étant substituée, elle nous donnera

$$\partial v = p\partial x - \frac{px\partial y}{y} = p\left(\frac{y\partial x - x\partial y}{y}\right).$$

Il faut donc que cette formule soit intégrable. Qu'on la réduise donc à cette forme:

$$\partial v = py\left(\frac{y\partial x - x\partial z}{yy}\right),$$

où posant $\frac{x}{y}=t$, pour avoir $py\,\partial t=\partial v$, il est clair que, pour que cette formule admit l'intégration, il faut absolument que py soit fonction de la seule variable t, et alors l'intégrale sera aussi une fonction de la même quantité t.

Employons dans la suite, pour marquer des fonctions quelconques, les caractères $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. de sorte que $\mathfrak{A}:t$, ou $\mathfrak{B}:t$, ou $\mathfrak{C}:t$ etc. nous représente une fonction quelconque de t. Outre cela nous nous servirons de la [illegible] assez généralement reçue, pour marquer les différentielles d'un ordre [illegible] savoir:

$$[illegible] = \partial t\,\mathfrak{A}':t,\ \partial\cdot\mathfrak{A}':t=\partial t\,\mathfrak{A}'':t,\ \partial\cdot\mathfrak{A}'':t=\partial t\,\mathfrak{A}''':t \text{ etc.}$$

Cela remarqué notre dernière équation intégrée donnera $v = \mathfrak{A}:t$, ou bien, à cause de $t = \frac{x}{y}$, nous aurons $v = \mathfrak{A}:\frac{x}{y}$; de sorte qu'on pourra prendre pour v une fonction quelconque de $\frac{x}{y}$; où il est bon de remarquer que toutes ces fonctions sont comprises sous le nom de fonctions homogènes de nulle dimension de x et y.

PROBLEME PRELIMINAIRE II

Trouver une fonction des deux variables x et y, qui soit v, telle qu'il y ait

$$nv = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y}.$$

SOLUTION

Posons, comme auparavant, $\partial v = p\,\partial x + q\,\partial y$, et puisque $p = \frac{\partial v}{\partial x}$ et $q = \frac{\partial v}{\partial y}$, nous aurons cette condition à remplir:

$$nv = px + qy.$$

Eliminons de ces deux équations la lettre q, en multipliant la première par y et l'autre par ∂y, et en ôtant la dernière de la première, nous aurons celle-ci:

$$y\,\partial v - nv\,\partial y = p(y\,\partial x - x\,\partial y);$$

où il faut chercher la fonction p, pour que cette équation devienne intégrable.

Pour rendre intégrable la première partie de cette équation, on n'a qu'à la diviser par y^{n+1}, d'où l'on tire

$$\frac{y\partial v - nv\partial y}{y^{n+1}} = \partial\cdot\frac{v}{y^n} = p\left(\frac{y\partial x - x\partial y}{y^{n+1}}\right).$$

Or puisque la formule $\frac{y\partial x - x\partial y}{y^2}$ est la différentielle de $\frac{x}{y}$, représentons nôtre équation sous cette forme:

$$\partial\cdot\frac{v}{y^n} = \frac{p}{y^{n-1}} \times \frac{y\partial x - x\partial y}{yy} = \frac{p}{y^{n-1}}\partial\cdot\frac{x}{y},$$

où il est évident que $\frac{p}{y^{n-1}}$ doit être fonction de $\frac{x}{y}$; et puisque l'intégrale sera par conséquent aussi une telle fonction, nous aurons, en intégrant cette équation,

$$\frac{v}{y^n} = \mathfrak{A}:\frac{x}{y};$$

d'où nous obtiendrons cette valeur pour la fonction cherchée:

$$v = y^n\,\mathfrak{A}:\frac{x}{y}\cdot$$

Puisque une fonction de $\frac{x}{y}$, étant multipliée par $\frac{x}{y}$, ou en général par $\frac{x^n}{y^n}$, demeure toujours fonction de $\frac{x}{y}$, au lieu de $\mathfrak{A}:\frac{x}{y}$ nous pourrons écrire $\frac{x^n}{y^n}\mathfrak{B}:\frac{x}{y}$, et partant la valeur trouvée pour v pourra aussi être exprimée par

$$v = x^n\,\mathfrak{B}:\frac{x}{y} \quad \text{ou bien } v = x^{n-1}y\,\mathfrak{B}:\frac{x}{y} \quad \text{ou bien } v = x^{n-2}y^2\,\mathfrak{B}:\frac{x}{y},$$

et ainsi de suite. Or on sait que toutes ces fonctions sont nommées homogènes, dont le nombre des dimensions est partout $= n$.

PROBLEME PRELIMINAIRE III

Trouver une fonction de deux variables x et y, qui soit v, telle qu'il y ait

$$nv = x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} + y^\lambda\mathfrak{A}:\frac{x}{y}\cdot$$

SOLUTION

Soit encore $dv = p\,\partial x + q\,\partial y$, pour avoir $p = \frac{\partial v}{\partial x}$ et $q = \frac{\partial v}{\partial y}$, et on aura cette condition à remplir:

$$nv = px + qy + y^\lambda\mathfrak{A}:\frac{x}{y}\cdot$$

Qu'on forme maintenant de ces deux équations celle-ci:

$$y\partial v - nv\partial y = p(y\partial x - x\partial y) - y^\lambda\partial y\,\mathfrak{A}:\frac{x}{y},$$

dont le premier membre deviendra intégrable en le divisant par y^{n+1}. Nous aurons donc

$$\partial\cdot\frac{v}{y^n} = p\frac{(y\,\partial x - x\partial y)}{y^{n+1}} - y^{\lambda-n-1}\partial y\,\mathfrak{A}:\frac{x}{y}\cdot$$

Pour résoudre cette équation mettons $\frac{x}{y} = t$, ou bien $x = yt$, et au lieu de $\mathfrak{A}:t$ écrivons T, de sorte que T soit une fonction donnée de t, et à cause de

$$\partial x = t\,\partial y + y\,\partial t$$

notre équation sera

$$\partial \cdot \frac{v}{y^n} = \frac{p\,\partial t}{y^{n-1}} - T y^{\lambda-n-1} \partial y\,.$$

Intégrons maintenant, en tant qu'il est permis, et puisque

$$\int T y^{\lambda-n-1} \partial y = \frac{y^{\lambda-n}}{\lambda-n} T - \int \frac{y^{\lambda-n}}{\lambda-n} T' \partial t\,,$$

en supposant $\partial T = T'\,\partial t$ nous aurons en intégrant

$$\frac{v}{y^n} = -\frac{y^{\lambda-n}}{\lambda-n} T + \int \partial t \left(\frac{p}{y^{n-1}} + \frac{y^{\lambda-n}}{\lambda-n} T' \right),$$

d'où l'on voit que la formule

$$\frac{p}{y^{n-1}} + \frac{y^{\lambda-n}}{\lambda-n} T'$$

doit être une fonction quelconque de t, que nous marquerons $\mathfrak{B}':t$, et partant nous aurons cette équation intégrale:

$$\frac{v}{y^n} = \mathfrak{B}:t - \frac{y^{\lambda-n}}{\lambda-n} T\,,$$

et de là

$$v = y^n \mathfrak{B}:t - \frac{y^{\lambda}}{\lambda-n} T\,.$$

Remettons à présent à la place de t sa valeur $\frac{x}{y}$ et $\mathfrak{A}:t$ au lieu de T, où il faut remarquer que le caractère $\mathfrak{A}$ marque une fonction donnée de $\frac{x}{y}$, puisque elle se trouve déjà dans l'équation différentielle donnée. Mais le caractère $\mathfrak{B}$ indiquera ici une fonction quelconque arbitraire de $\frac{x}{y}$, qui est introduite dans les intégrations ordinaires. Par conséquent nous aurons pour la solution de notre problème la valeur suivante de la fonction v, savoir

$$v = y^n \mathfrak{B}:\frac{x}{y} - \frac{y^{\lambda}}{\lambda-n} \mathfrak{A}:\frac{x}{y}\,.$$

Ici on demandera peut-être quelle sera la valeur de v, au cas que l'exposant λ seroit égal à n, puisque alors le dernier membre de notre équation deviendroit infini? Pour écarter cette difficulté mettons $\lambda = n + \omega$, en marquant par ω une quantité infiniment petite, et nous aurons

$$y^\lambda = y^n \cdot y^\omega = y^n(1 + \omega l y),$$

ce qui nous donnera $v = y^n \mathfrak{B}:\frac{x}{y} - \frac{y^n(1+\omega l y)}{\omega}\mathfrak{A}:\frac{x}{y}$. Maintenant puisque $\mathfrak{B}$ marque une fonction arbitraire, il sera permis de mettre à la place de $\mathfrak{B}:\frac{x}{y}$ cette formule:

$$\frac{1}{\omega}\mathfrak{A}:\frac{x}{y} + \mathfrak{C}:\frac{x}{y},$$

où $\mathfrak{C}$ marque une fonction arbitraire quelconque, et en substituant ces valeurs les membres infinis se détruiront et l'intégrale cherchée pour le cas $\lambda = n$ sera

$$v = y^n \mathfrak{C}:\frac{x}{y} - y^n l y \mathfrak{A}:\frac{x}{y}\cdot$$

Nous serons donc en état de résoudre maintenant le problème suivant.

PROBLEME I

Trouver l'intégrale complette de cette équation différentielle:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0,$$

ou bien chercher la nature de la fonction z.

SOLUTION

Ici nous avons donc $P = 0$, et le premier problème préliminaire nous fournira d'abord l'intégrale cherchée, puisqu'on n'a qu'à écrire z au lieu de v, et partant notre intégrale complette sera $z = \mathfrak{A}:\frac{x}{y}$. Ou bien on pourra prendre pour z une fonction quelconque homogène de nulle dimension de x et y.

Qu'on prenne, par exemple,

$$z = \frac{xx - yy}{xx + yy},$$

et on aura

$$\frac{\partial z}{\partial x} = \frac{4xyy}{(xx+yy)^2} \quad \text{et} \quad \frac{\partial z}{\partial y} = -\frac{4xxy}{(xx+yy)^2},$$

d'où il devient évidemment

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

De la même manière, en prenant $z = \frac{x+y}{\sqrt{xx+yy}}$, on aura

$$\frac{\partial z}{\partial x} = \frac{yy-xy}{(xx+yy)^{\frac{3}{2}}} \quad \text{et} \quad \frac{\partial z}{\partial y} = \frac{xx-xy}{(xx+yy)^{\frac{3}{2}}},$$

et de là il s'ensuit ouvertement

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

PROBLEME II

Trouver l'intégrale complette de cette équation différentielle du second degré:

$$xx\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + yy\frac{\partial\partial z}{\partial y^2} = 0.$$

SOLUTION

On suppose donc ici que $Q = 0$, et partant, puisque nous avons trouvé ci-dessus

$$Q = x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P,$$

nous aurons à résoudre cette équation différentielle du premier degré: $x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} = P$ dont l'intégrale se trouve par le second problème préliminaire, en mettant P au lieu de v et $n = 1$, d'où l'on tire $P = y\mathfrak{A}:\frac{x}{y}$, où $\mathfrak{A}$ marque une fonction quelconque. Mettons à présent au lieu de P sa valeur, et nous aurons à résoudre cette équation différentielle du premier degré:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = y\mathfrak{A}:\frac{x}{y}.$$

Cette équation étant comparée avec le troisième problème préliminaire nous donne $n = 0$, $\lambda = 1$ et $v = z$; par conséquent l'intégrale complette cherchée de notre équation sera

$$z = \mathfrak{B} : \frac{x}{y} - y\mathfrak{A} : \frac{x}{y}$$ [1],

ou bien, puisque les deux fonctions sont arbitraires, on pourra mettre

$$z = \mathfrak{A} : \frac{x}{y} + y\mathfrak{B} : \frac{x}{y},$$

qui renferme par conséquent deux fonctions arbitraires, comme la nature des équations différentielles du second ordre l'exige.

PROBLEME III

Trouver l'intégrale complette de cette équation différentielle du troisième degré :

$$x^3 \frac{\partial^3 z}{\partial x^3} + 3xxy \frac{\partial^3 z}{\partial x^2 \partial y} + 3xyy \frac{\partial^3 z}{\partial x \partial y^2} + y^3 \frac{\partial^3 z}{\partial y^3} = 0.$$

SOLUTION

Il s'agit donc ici de rendre $R = 0$, et en mettant pour R sa valeur indiquée ci-dessus, nous aurons à résoudre cette équation:

$$x \frac{\partial Q}{\partial x} + y \frac{\partial Q}{\partial y} - 2Q = 0,$$

qui, étant comparée avec celle du second problème préliminaire, donne $v = Q$ et $n = 2$, donc son intégrale complette est

$$Q = y^2 \mathfrak{A} : \frac{x}{y}.$$

Maintenant ayant

$$Q = x \frac{\partial P}{\partial x} + y \frac{\partial P}{\partial y} - P,$$

1) Si l'on désignait par $\mathfrak{A} : \frac{x}{y}$ la même fonction que dans la formule précédente, il faudrait $+ y\mathfrak{A} : \frac{x}{y}$. On doit admettre que les notations $\mathfrak{A} : \frac{x}{y}$, $\mathfrak{B} : \frac{x}{y}$ ne sont que des abréviations de la phrase: fonction arbitraire de $\frac{x}{y}$. Une même lettre peut désigner deux fonctions distinctes, soit dans deux formules différentes, soit dans une même formule. Il n'y a en particulier à tenir compte ni des signes, ni des coefficients qui précèdent ces fonctions. H. D.

nous aurons à résoudre l'équation

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P = y^2\mathfrak{A}:\frac{x}{y},$$

qui, étant comparée avec celle du troisième problème préliminaire, donne $v = P$, $n = 1$, $\lambda = 2$, ce qui étant substitué donne l'intégrale suivante

$$P = y\mathfrak{B}:\frac{x}{y} - y^2\mathfrak{A}:\frac{x}{y},$$

ou bien, puisque les fonctions sont arbitraires, on aura

$$P = y^2\mathfrak{A}:\frac{x}{y} + y\mathfrak{B}:\frac{x}{y}\cdot$$

Enfin donc puisque

$$P = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y},$$

nous aurons cette équation à résoudre:

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = y^2\mathfrak{A}:\frac{x}{y} + y\cdot\mathfrak{B}:\frac{x}{y},$$

qui, comparée avec l'équation du troisième problème préliminaire, nous fournit $v = z$, $n = 0$, et pour λ nous aurons deux valeurs différentes, ou $\lambda = 2$, ou $\lambda = 1$; car il est évident que l'un et l'autre pourra être traité de la même manière; par conséquent l'intégrale complette de l'équation proposée sera

$$z = \mathfrak{B}:\frac{x}{y} - y\mathfrak{A}:\frac{x}{y} - y^2\mathfrak{A}:\frac{x}{y},$$

ou bien en changeant les caractères, signes des fonctions arbitraires, il y aura

$$z = \mathfrak{A}:\frac{x}{y} + y\mathfrak{B}:\frac{x}{y} + y^2\mathfrak{C}:\frac{x}{y}\cdot$$

PROBLEME IV

Trouver l'intégrale complette de cette équation différentielle du quatrième degré:

$$0 = x^4\frac{\partial^4 z}{\partial x^4} + 4x^3y\frac{\partial^4 z}{\partial x^3\partial y} + 6xxyy\frac{\partial^4 z}{\partial x^2\partial y^2} + 4xy^3\frac{\partial^4 z}{\partial x\partial y^3} + y^4\frac{\partial^4 z}{\partial y^4}\cdot$$

SOLUTION

On aura donc ici $S = 0$, ou bien

$$x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} - 3R = 0,$$

ce qui, comparé avec le second problème préliminaire, fournit $v = R$ et $n = 3$ et partant

$$R = y^3\mathfrak{A}:\frac{x}{y}.$$

Mettant donc au lieu de R sa valeur, il faudra résoudre cette équation

$$x\frac{\partial Q}{\partial y} + y\frac{\partial Q}{\partial y} - 2Q = y^3\mathfrak{A}:\frac{x}{y},$$

qui, comparée avec le troisième préliminaire, à cause de $v = Q$, $n = 2$ et $\lambda = 3$, donne

$$Q = y^2\mathfrak{B}:\frac{x}{y} + y^3\mathfrak{A}:\frac{x}{y},$$

ou bien

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} - P = y^2\mathfrak{B}:\frac{x}{y} + y^3\mathfrak{A}:\frac{x}{y};$$

où il y a par conséquent $v = P$, $n = 1$ et $\lambda = 2$ ou 3, d'où l'on tire

$$P = y\mathfrak{B}:\frac{x}{y} + y^2\mathfrak{A}:\frac{x}{y} + y^3\mathfrak{A}:\frac{x}{y},$$

ou bien

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = y\mathfrak{A}:\frac{x}{y} + y^2\mathfrak{B}:\frac{x}{y} + y^3\mathfrak{C}:\frac{x}{y},$$

qui comparaison faite donne $v = z$, $n = 0$ et $\lambda = 1$ ou 2, ou 3, ce qui donne

$$z = \mathfrak{B}:\frac{x}{y} - y\mathfrak{A}:\frac{x}{y} - y^2\mathfrak{A}:\frac{x}{y} - y^3\mathfrak{A}:\frac{x}{y}\ ^{1)},$$

ou bien

$$z = \mathfrak{A}:\frac{x}{y} + y\mathfrak{B}:\frac{x}{y} + y^2\mathfrak{C}:\frac{x}{y} + y^3\mathfrak{D}:\frac{x}{y}.$$

1) Voir la note de la p. 360. H. D.

PROBLEME V GENERAL

Trouver l'intégrale complette de cette équation différentielle du degré n^{ieme}:

$$x^n \frac{\partial^n z}{\partial x^n} + \frac{n}{1} x^{n-1} y \frac{\partial^n z}{\partial x^{n-1} \partial y} + \frac{n}{1} \cdot \frac{n-1}{2} x^{n-2} y^2 \frac{\partial^n z}{\partial x^{n-2} \partial y^2} + \text{etc.}$$

SOLUTION

Ici il est facile à voir qu'en faisant les opérations successivement comme dans les problèmes précédens on parviendra enfin à cette intégrale complette:

$$z = \mathfrak{A} : \frac{x}{y} + y \mathfrak{B} : \frac{x}{y} + y^2 \mathfrak{C} : \frac{x}{y} + \cdots + y^{n-1} \mathfrak{N} : \frac{x}{y},$$

où le nombre des fonctions arbitraires est $= n$, et partant égal au degré de l'équation proposée; d'où l'on voit que l'intégrale de chaque degré renferme toutes les intégrales de tous les degrés inférieurs, et outre cela encore un terme qui apartient exclusivement au degré proposé.

Voilà donc les intégrations de toutes ces équations différentielles

$$1^0.\ P = 0, \quad 2^0.\ Q = 0, \quad 3^0.\ R = 0, \quad 4^0.\ S = 0 \quad \text{etc.}$$

en assignant à chacune de ces lettres les valeurs qui leur ont été données au commencement, et la méthode dont nous nous sommes servis demande pour chaque cas autant d'intégrations que le degré du différentiel indique. Or un jeune Géomètre, en faisant les calculs précédens, a observé: que toutes ces solutions pourront être exécutées plus facilement moyennant une seule intégration, et cette méthode a encore ce grand avantage sur celle dont nous nous sommes servis jusqu'ici, qu'elle s'étend aussi à l'intégration des équations différentielles composées et comprises dans cette forme générale:

$$Az + BP + CQ + DR + ES + \text{etc.} = 0,$$

où tous les degrés des différentielles se trouvent joints ensemble, et où les coefficiens constans A, B, C, D etc. peuvent être pris à volonté. Et la résolution de tous ces cas se peut toujours tirer du seul problème préliminaire second, qui donne pour l'équation différentielle $x \frac{\partial v}{\partial x} + y \frac{\partial v}{\partial y} - nv = 0$ cette intégrale complette: $y^n \mathfrak{A} : \frac{x}{y}$. Pour éclaircir cette nouvelle méthode, nous ajouterons les problèmes suivans.

PROBLEME I

Trouver l'intégrale complette de cette équation différentielle du premier degré:

$$Az + BP = 0,$$

ou bien

$$Az + B\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) = 0.$$

SOLUTION

Pour cet effet mettons dans le problème préliminaire $v = az$, pour avoir cette équation:

$$a\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - naz = 0,$$

dont l'intégrale est $z = y^n \mathfrak{A} : \frac{x}{y}$. Maintenant au lieu de

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}$$

mettons sa valeur assignée P, et l'équation que nous venons d'intégrer sera $aP - naz = 0$, qui, comparée avec la proposée $Az + BP$, donne

$$A = -na \quad \text{et} \quad B = a,$$

par conséquent

$$a = B \quad \text{et} \quad A = -nB, \quad \text{ou bien} \quad A + nB = 0.$$

En tirant de cette équation la valeur de

$$n = -\frac{A}{B},$$

l'intégrale de l'équation proposée sera $y^n \mathfrak{A} : \frac{x}{y}$. Cette solution ne renferme rien qui n'auroit pu être fait par la méthode précédente, mais le problème suivant mettra dans tout son jour le prix de la nouvelle méthode.

PROBLEME II

Trouver l'intégrale complette de cette équation différentielle du second degré:

$$Az + BP + CQ = 0.$$

SOLUTION

Pour résoudre cette équation supposons dans le problème préliminaire

$$v = az + bP,$$

pour avoir cette intégrale

$$az + bP = y^n \mathfrak{A} : \frac{x}{y},$$

qui convient donc avec cette équation

$$a\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right) - naz + b\left(x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}\right) - nbP = 0.$$

Mettons à présent dans cette équation, au lieu de

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y},$$

sa valeur absolue tirée des formules supposées au commencement, laquelle est P, et au lieu de la formule

$$x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y}$$

mettons cette valeur absolue $Q + P$, et nous aurons cette équation:

$$aP + bQ + bP - naz - nbP = 0,$$

ou bien

$$-naz + (a + b - nb)P + bQ = 0,$$

qui, étant comparée avec la forme supposée

$$Az + BP + CQ = 0,$$

nous donne pour les lettres a et b les valeurs suivantes:

$$b = C, \; a = B - C + nC, \quad \text{et} \quad 0 = A + nB + n(n-1)C,$$

d'où il faut tirer la valeur de n.

Or puisque cette dernière équation est du second degré, elle aura deux racines, qui soyent α et β, dont chacune nous donnera des valeurs particulières pour a et b, qui sont:

$$\begin{array}{ll} n = \alpha, & n = \beta, \\ a = B + (\alpha - 1)C, & a = B + (\beta - 1)C, \\ b = C, & b = C; \end{array}$$

de là nous aurons deux équations intégrales

$$(B+(\alpha-1)C)z+CP=y^{\alpha}\mathfrak{A}:\frac{x}{y},$$

$$(B+(\beta-1)C)z+CP=y^{\beta}\mathfrak{B}:\frac{x}{y}.$$

Maintenant de ces deux équations on n'a qu'à chasser la lettre P, ce qui se fait en prenant leur différence, ce qui donne

$$(\alpha-\beta)Cz=y^{\alpha}\mathfrak{A}:\frac{x}{y}-y^{\beta}\mathfrak{B}:\frac{x}{y};$$

et puisque les fonctions sont absolument arbitraires, on pourra représenter l'intégrale sous cette forme:

$$z=y^{\alpha}\mathfrak{A}:\frac{x}{y}+y^{\beta}\mathfrak{B}:\frac{x}{y}.$$

COROLLAIRE

De là se déduit aisément l'intégrale de l'équation $Q=0$ que nous avons traitée ci-dessus; on n'a qu'à supposer $A=0$ et $B=0$ et $C=1$, et alors l'équation pour le nombre n devient $n(n-1)=0$, dont les racines sont $n=0$ et $n=1$, par conséquent $\alpha=0$ et $\beta=1$, et partant l'intégrale de ce cas sera

$$z=\mathfrak{A}:\frac{x}{y}+y\mathfrak{B}:\frac{x}{y}.$$

PROBLEME III

Trouver l'intégrale complette de cette équation différentielle du troisième degré:

$$Az+BP+CQ+DR=0.$$

SOLUTION

Pour parvenir à la solution de ce problème, supposons dans le second problème préliminaire $v=az+bP+cQ$, et l'intégration nous fournit d'abord cette équation: $az+bP+cQ=y^{n}\,\mathfrak{A}:\frac{x}{y}$, et cette intégrale convient à l'équation différentielle suivante:

$$\left.\begin{aligned} &a\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)-naz\\ +\ &b\left(x\frac{\partial P}{\partial x}+y\frac{\partial P}{\partial y}\right)-nbP\\ +\ &c\left(x\frac{\partial Q}{\partial x}+y\frac{\partial Q}{\partial y}\right)-ncQ \end{aligned}\right\}=0.$$

Maintenant au lieu des formules différentielles mettons leurs valeurs finies, et nous parviendrons à cette équation:

$$aP+b(P+Q)+c(R+2Q)-naz-nbP-ncQ=0$$

qui se réduit à cette forme:

$$-naz+(a+b(1-n))P+(b+(2-n)c)Q+cR=0,$$

qui, étant comparée avec la proposée, nous fournit les équations de condition suivantes:

$$A=-na,\quad B=a+b(1-n),\quad C=b+(2-n)c,\quad D=c.$$

Ayant donc de la dernière $D=c$, la troisième nous donnera

$$b=C+(n-2)D;$$

ensuite la seconde équation nous fournit

$$a=B+(n-1)C+(n-1)(n-2)D,$$

et cette valeur substituée dans la première donne cette valeur finale:

$$A+nB+n(n-1)C+n(n-1)(n-2)D=0,$$

qui étant du troisième degré renferme trois racines, qui soyent α, β, γ. Chacune de ces racines nous donnera des valeurs particulières pour les lettres a, b, c, qui étant rapportées à la racine α[1]), supposons que pour la racine β on ait a', b', c', et qu'à la racine γ répondissent celles-ci: a'', b'', c'', chacun de ces cas nous fournira donc une équation intégrale particulière, et ces équations seront

$$\begin{aligned} az+bP+cQ&=y^{\alpha}\mathfrak{A}:\frac{x}{y}\,,\\ a'z+b'P+c'Q&=y^{\beta}\mathfrak{B}:\frac{x}{y}\,,\\ a''z+b''P+c''Q&=y^{\gamma}\mathfrak{C}:\frac{x}{y}\,. \end{aligned}$$

1) La phrase „qui étant rapportées à la racine α“ doit être interprétée: Soient a, b, c les valeurs correspondant à la racine α. H. D.

A présent il sera facile de trouver pour chacune de ces équations certains multiplicateurs tels, qu'en ajoutant les produits ensemble les quantités P et Q soyent détruites; et puisque ces multiplicateurs ne changent pas la nature des fonctions arbitraires, on parviendra par ce moyen à cette équation finale:

$$z = y^{\alpha}\mathfrak{A}:\frac{x}{y} + y^{\beta}\mathfrak{B}:\frac{x}{y} + y^{\gamma}\mathfrak{C}:\frac{x}{y}$$

qui exprime l'intégrale complette de notre équation différentielle proposée.

COROLLAIRE

Pour tirer de là l'intégrale de l'équation $R = 0$, on n'a qu'à mettre

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 1,$$

et alors l'équation cubique pour le nombre n deviendra

$$n(n-1)(n-2) = 0,$$

dont les trois racines sont ouvertement 0, 1, 2, de sorte que $\alpha = 0$, $\beta = 1$, $\gamma = 2$; d'où nous tirons l'intégrale cherchée pour ce cas

$$z = \mathfrak{A}:\frac{x}{y} + y\mathfrak{B}:\frac{x}{y} + y^2\mathfrak{C}:\frac{x}{y},$$

qui convient parfaitement avec celle qui a été trouvée ci-dessus.

PROBLEME IV

Trouver l'intégrale complette de cette équation différentielle du quatrième degré :

$$Az + BP + CQ + DR + ES = 0.$$

SOLUTION

Pour résoudre cette question mettons dans le second problème préliminaire

$$v = az + bP + cQ + dR,$$

et nous aurons d'abord cette intégrale:

$$az + bP + cQ + dR = y^n\mathfrak{A}:\frac{x}{y},$$

qui conviendra donc a cette équation différentielle:

$$\left.\begin{array}{r} a\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)-naz \\ +\,b\left(x\dfrac{\partial P}{\partial x}+y\dfrac{\partial P}{\partial y}\right)-nbP \\ +\,c\left(x\dfrac{\partial Q}{\partial x}+y\dfrac{\partial Q}{\partial y}\right)-ncQ \\ +\,d\left(x\dfrac{\partial R}{\partial x}+y\dfrac{\partial R}{\partial y}\right)-ndR \end{array}\right\}=0.$$

Maintenant qu'on écrive au lieu des formules différentielles leurs valeurs finies, pour arriver à cette équation:

$$\left.\begin{array}{l} +\,aP \qquad\qquad -naz \\ +\,b\,(Q+\ \ P)-nbP \\ +\,c\,(R+2Q)-ncQ \\ +\,d\,(S+3R)-ndR \end{array}\right\}=0,$$

dont les termes étant rangés donneront

$$-naz+(a+b(1-n))P+(b+c(2-n))Q+(c+d(3-n))R+dS=0.$$

Il ne reste donc qu'à rendre identique cette forme avec la proposée, ce qui produit les cinq égalités suivantes:

$1^{0}.\,A=-na,\ 2^{0}.\,B=a+(1-n)b,\ 3^{0}.\,C=b+(2-n)c,\ 4^{0}.\,D=c+(3-n)d,\ 5^{0}.\,E=d.$

La dernière nous donne d'abord $d=E$, la quatrième fournit

$$c=D+(n-3)E,$$

ensuite de la troisième nous tirons

$$b=C+(n-2)D+(n-2)(n-3)E,$$

la seconde fournit

$$a=B+(n-1)C+(n-1)(n-2)D+(n-1)(n-2)(n-3)E,$$

enfin la première nous conduit à cette équation pour la détermination du nombre n:

$$A+nB+n(n-1)C+n(n-1)(n-2)D+n(n-1)(n-2)(n-3)E=0.$$

Cette dernière équation étant du quatrième degré soyent α, β, γ, δ les quatre valeurs du nombre n, dont chacune produira pour les lettres a, b, c, d des valeurs particulières. Mettons donc pour la racine α ces mêmes lettres a, b, c, d, pour la racine β: a', b', c', d', pour γ: a'', b'', c'', d'', et pour δ: a''', b''', c''', d'''; et alors nous aurons quatre formes différentes de l'équation intégrale trouvée qui seront:

$$a\ z + b\ P + c\ Q + d\ R = y^\alpha \mathfrak{A} : \frac{x}{y},$$

$$a'\ z + b'\ P + c'\ Q + d'\ R = y^\beta \mathfrak{B} : \frac{x}{y},$$

$$a''\ z + b''\ P + c''\ Q + d''\ R = y^\gamma \mathfrak{C} : \frac{x}{y},$$

$$a'''z + b'''P + c'''Q + d'''R = y^\delta \mathfrak{D} : \frac{x}{y}.$$

Après avoir trouvé ces quatre équations, il est facile d'en éliminer les trois quantités P, Q, R, de sorte qu'il ne restera à la gauche que la seule quantité z, et puisque les fonctions à la droite, étant multipliées par certaines constantes, ne changent point de nature, on en tirera cette équation finale:

$$z = y^\alpha \mathfrak{A} : \frac{x}{y} + y^\beta \mathfrak{B} : \frac{x}{y} + y^\gamma \mathfrak{C} : \frac{x}{y} + y^\delta \mathfrak{D} : \frac{x}{y},$$

où il est bon de remarquer que pour trouver cette équation nous n'avons pas eu besoin de trouver les valeurs de a, b, c, d, ni même les multiplicateurs, pour l'élimination des quantités P, Q, R.

PROBLEME V GENERAL

Trouver l'intégrale complette de cette équation différentielle d'un degré quelconque:

$$Az + BP + CQ + DR + \text{etc.} = 0.$$

SOLUTION

Toute la solution de cette question se reduit à l'équation pour déterminer toutes les valeurs du nombre n; et il est clair par les problèmes précédens, que cette équation aura la forme

$$A + nB + n(n-1)C + n(n-1)(n-2)D + \text{etc.} = 0,$$

qui montera au même degré auquel se rapporte l'équation différentielle proposée; et partant le nombre n aura autant de valeurs, que nous marquerons par les lettres α, β, γ, δ etc. et alors l'intégrale complette de l'équation proposée sera

$$z = y^\alpha \mathfrak{A} : \frac{x}{y} + y^\beta \mathfrak{B} : \frac{x}{y} + y^\gamma \mathfrak{C} : \frac{x}{y} + y^\delta \mathfrak{D} : \frac{x}{y} + \text{etc.},$$

qui comprend autant de fonctions arbitraires que l'ordre de la différentielle demande.

Ici il est bon de remarquer, que puisque les deux variables x et y entrent également dans le calcul, au lieu des puissances y^α, y^β, y^γ etc. on pourra aussi mettre de semblables puissances de x, savoir x^α, x^β, x^γ etc. Et en effet, si nous considérons la formule $y^\alpha \mathfrak{A} : \frac{x}{y}$, puisque $\frac{x^\lambda}{y^\lambda}$ est aussi une fonction de $\frac{x}{y}$, au lieu de $\mathfrak{A} : \frac{x}{y}$ on pourra mettre $\frac{x^\lambda}{y^\lambda} \mathfrak{F} : \frac{x}{y}$; et alors nous aurons $x^\lambda y^{\alpha-\lambda} \mathfrak{F} : \frac{x}{y}$. Donc prenant $\lambda = \alpha$, au lieu de la formule $y^\alpha \mathfrak{A} : \frac{x}{y}$ on pourra mettre $x^\alpha \mathfrak{A} : \frac{x}{y}$; et il est aussi clair qu'on pourroit écrire en général $x^\mu y^\nu \mathfrak{A} : \frac{x}{y}$, pourvu que la somme des exposans μ et ν fut égale à α, c'est-à-dire $\mu + \nu = \alpha$.

Cette solution n'aura donc aucune difficulté, tant que les valeurs de l'exposant n, que nous supposons être α, β, γ, δ etc. sont toutes réelles et inégales entre elles. Mais dans le cas où quelques unes de ces valeurs sont ou imaginaires ou égales entre elles, il faut recourir à certaines réductions pour rendre l'intégrale réelle dans le premier cas; or pour l'autre cas il faut que le nombre nécessaire des fonctions arbitraires reste non-diminué, sans quoi l'intégrale ne seroit plus complette.

Pour lever toutes ces difficultés commençons par considérer le cas où deux valeurs de n se trouvent imaginaires, savoir α et β, et on sait que ces deux valeurs se réduiront toujours à ces formes:

$$\alpha = \mu + \nu\sqrt{-1} \quad \text{et} \quad \beta = \mu - \nu\sqrt{-1},$$

et partant les termes de l'intégrale, qui dépendent de ces valeurs, seront

$$y^{\mu+\nu\sqrt{-1}} \mathfrak{A} : \frac{x}{y} \quad \text{et} \quad y^{\mu-\nu\sqrt{-1}} \mathfrak{B} : \frac{x}{y};$$

et pour les réduire à la réalité supposons

$$\mathfrak{A}:\frac{x}{y}=\mathfrak{F}:\frac{x}{y}+\mathfrak{G}:\frac{x}{y} \quad \text{et} \quad \mathfrak{B}:\frac{x}{y}=\mathfrak{F}:\frac{x}{y}-\mathfrak{G}:\frac{x}{y},$$

et à présent ces deux termes en question se réduiront à cette forme:

$$y^{\mu}(y^{\nu\sqrt{-1}}+y^{-\nu\sqrt{-1}})\,\mathfrak{F}:\frac{x}{y}+y^{\mu}(y^{\nu\sqrt{-1}}-y^{-\nu\sqrt{-1}})\,\mathfrak{G}:\frac{x}{y}\,.$$

Mettons ici dans les puissances imaginaires e^{ly} au lieu de y, en prenant pour e le nombre dont le logarithme hyperbolique est 1, et la première formule $y^{\nu\sqrt{-1}}+y^{-\nu\sqrt{-1}}$ deviendra

$$=e^{\nu ly\sqrt{-1}}+e^{-\nu ly\sqrt{-1}},$$

et l'autre $y^{\nu\sqrt{-1}}-y^{-\nu\sqrt{-1}}$ deviendra

$$=e^{\nu ly\sqrt{-1}}-e^{-\nu ly\sqrt{-1}}.$$

Or on sait par les réductions connues que

$$e^{v\sqrt{-1}}+e^{-v\sqrt{-1}}=2\cos.v \quad \text{et} \quad e^{v\sqrt{-1}}-e^{-v\sqrt{-1}}=2\sqrt{-1}\sin.v.$$

Donc puisque $v=\nu ly$, la forme de nos deux termes sera

$$y^{\mu}\cdot 2\cos.v\cdot\mathfrak{F}:\frac{x}{y}+y^{\mu}\cdot 2\sqrt{-1}\cdot\sin.v\cdot\mathfrak{G}:\frac{x}{y};$$

où l'on peut omettre les coefficiens constans tant réels qu'imaginaires. Nous aurons donc, au lieu des deux termes proposés, ceux-ci:

$$y^{\mu}\cos.(\nu ly)\cdot\mathfrak{F}:\frac{x}{y}+y^{\mu}\sin.(\nu ly)\cdot\mathfrak{G}:\frac{x}{y},$$

toutes les fois que

$$\alpha=\mu+\nu\sqrt{-1} \quad \text{et} \quad \beta=\mu-\nu\sqrt{-1}.$$

De là il est clair que lorsque le nombre des valeurs imaginaires de n est 4, 6, 8, 10 etc. puisque chaque couple se réduit toujours à ces deux formules $\mu+\nu\sqrt{-1}$ et $\mu-\nu\sqrt{-1}$, la réduction se pourra toujours faire de la même manière.

Pour en donner un exemple prenons le cas où l'équation, pour déterminer le nombre n, devient $1+nn=0$, qui appartient au second degré, où nous avions trouvé

$$A+nB+n(n-1)C=0,$$

il faudra prendre $A = B = C = 1$, de sorte que l'équation différentielle à intégrer sera pour ce cas

$$z + P + Q = 0,$$

ou bien en la développant:

$$z + x\frac{\partial z}{\partial x} + y\frac{\partial x}{\partial y} + xx\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + yy\frac{\partial\partial z}{\partial y^2} = 0.$$

Et puisque pour n nous aurons ces valeurs $\alpha = \sqrt{-1}$, $\beta = -\sqrt{-1}$, et partant $\mu = 0$ et $\nu = 1$, nous en déduisons d'abord

$$z = \cos. ly\, \mathfrak{F} : \frac{x}{y} + \sin. ly\, \mathfrak{G} : \frac{x}{y}.$$

Pour mieux éclaircir ce cas-ci prenons

$$\mathfrak{F} : \frac{x}{y} = 0 \quad \text{et} \quad \mathfrak{G} : \frac{x}{y} = \frac{x}{y},$$

de sorte qu'une intégrale particulière sera

$$z = \frac{x}{y} \sin. ly,$$

d'où nous tirons

$$\frac{\partial z}{\partial x} = \frac{1}{y}\cdot \sin. ly \quad \text{et} \quad \frac{\partial z}{\partial y} = -\frac{x}{yy}\cdot \sin. ly + \frac{x}{yy}\cdot \cos. ly,$$

et ensuite

$$\frac{\partial\partial z}{\partial x^2} = 0 \quad \text{et} \quad \frac{\partial\partial z}{\partial x\partial y} = -\frac{1}{yy}\cdot \sin. ly + \frac{1}{yy}\cdot \cos. ly \quad \text{et} \quad \frac{\partial\partial z}{\partial y^2} = \frac{x}{y^3}\cdot \sin. ly - \frac{3x}{y^3}\cdot \cos. ly.$$

Ces valeurs, étant substituées dans l'équation $z + P + Q = 0$, donneront

$$z = \frac{x}{y} \sin. ly,$$

$$P = \frac{x}{y} \cos. ly,$$

$$Q = -\frac{x}{y} \sin. ly - \frac{x}{y} \cos. ly,$$

dont la somme donne $z + P + Q = 0$.

Passons au cas où deux ou plusieurs valeurs de n deviennent égales entr'elles. Supposons d'abord que $\beta = \alpha$, et dans la forme intégrale trouvée les deux premiers termes

$$y^\alpha \mathfrak{A}:\frac{x}{y} + y^\beta \mathfrak{B}:\frac{x}{y}$$

se réduiroient à une seule fonction, et partant l'intégrale ne seroit plus complette. Pour remplir ce nombre posons $\beta = \alpha + \omega$, en prenant ω pour un infiniment-petit, et à cause de $y^\beta = y^\alpha \cdot y^\omega$ et de $y^\omega = 1 + \omega l y$, on aura

$$y^\beta = y^\alpha + \omega y^\alpha l y,$$

d'où les deux premiers termes deviendront

$$y^\alpha \mathfrak{A}:\frac{x}{y} + y^\alpha \mathfrak{B}:\frac{x}{y} + \omega y^\alpha l y \mathfrak{B}:\frac{x}{y};$$

où au lieu des deux premiers termes on peut écrire simplement $y^\alpha \mathfrak{A}:\frac{x}{y}$, et $\mathfrak{B}:\frac{x}{y}$ au lieu de $\omega \mathfrak{B}:\frac{x}{y}$; de sorte qu'au lieu des deux premiers termes nous aurons à présent:

$$y^\alpha \mathfrak{A}:\frac{x}{y} + y^\alpha l y \mathfrak{B}:\frac{x}{y}.$$

Pour donner un exemple de ce cas, supposons que l'équation pour déterminer le nombre n, soit $nn = 0$, et cette équation appartiendra au second degré, pour lequel nous avions en général

$$A + nB + n(n-1)C = 0,$$

où il faudra mettre $A = 0$, $B = 1$ et $C = 1$, de sorte que l'équation différentielle à intégrer sera $P + Q = 0$, ou bien

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} + xx\frac{\partial\partial z}{\partial x^2} + 2xy\frac{\partial\partial z}{\partial x\partial y} + yy\frac{\partial\partial z}{\partial y^2} = 0.$$

Ayant donc pour la résolution de cette équation $nn = 0$, les deux valeurs égales de n seront $\alpha = 0$ et $\beta = 0$; par conséquent l'intégrale complette cherchée de cette équation est

$$z = \mathfrak{A}:\frac{x}{y} + l y \mathfrak{B}:\frac{x}{y};$$

où il vaudra la peine de faire voir comment cette valeur satisfait en général à l'équation proposée. Pour cet effet nous différentierons ces formules selon la règle établie ci-dessus $\partial \cdot \mathfrak{A}:v = \partial v \mathfrak{A}':v$ et $\partial \cdot \mathfrak{A}':v = \partial v \mathfrak{A}'':v$, et nous trouverons:

$$\frac{\partial z}{\partial x}=\frac{1}{y}\mathfrak{A}':\frac{x}{y}+\frac{ly}{y}\mathfrak{B}':\frac{x}{y},$$
$$\frac{\partial z}{\partial y}=-\frac{x}{yy}\mathfrak{A}':\frac{x}{y}+\frac{1}{y}\mathfrak{B}:\frac{x}{y}-\frac{xly}{yy}\mathfrak{B}':\frac{x}{y},$$
$$\frac{\partial\partial z}{\partial x^2}=\frac{1}{yy}\mathfrak{A}'':\frac{x}{y}+\frac{ly}{yy}\mathfrak{B}'':\frac{x}{y},$$
$$\frac{\partial\partial z}{\partial x\,\partial y}=-\frac{1}{yy}\mathfrak{A}':\frac{x}{y}-\frac{x}{y^3}\mathfrak{A}'':\frac{x}{y}+\frac{1}{yy}\mathfrak{B}':\frac{x}{y}-\frac{ly}{yy}\mathfrak{B}':\frac{x}{y}-\frac{xly}{y^3}\mathfrak{B}'':\frac{x}{y},$$
$$\frac{\partial\partial z}{\partial y^2}=+\frac{2x}{y^3}\mathfrak{A}':\frac{x}{y}+\frac{xx}{y^4}\mathfrak{A}'':\frac{x}{y}-\frac{1}{yy}\mathfrak{B}:\frac{x}{y}-\frac{2x}{y^3}\mathfrak{B}':\frac{x}{y}+\frac{2xly}{y^3}\mathfrak{B}':\frac{x}{y}+\frac{x^2ly}{y^4}\mathfrak{B}'':\frac{x}{y},$$

d'où nous tirons la formule suivante:

$$P=\frac{x}{y}\mathfrak{A}':\frac{x}{y}+\frac{xly}{y}\mathfrak{B}':\frac{x}{y}-\frac{x}{y}\mathfrak{A}':\frac{x}{y}+\mathfrak{B}:\frac{x}{y}-\frac{xly}{y}\mathfrak{B}':\frac{x}{y}$$

ou bien:

$$P=\mathfrak{B}:\frac{x}{y}.$$

De la même manière on trouvera $Q=-\mathfrak{B}:\frac{x}{y}$, d'où il s'ensuit ouvertement $P+Q=0$. Ce développement paroît d'autant plus nécessaire qu'on ne trouve nulle part des règles particulières pour différentier les fonctions à deux variables.

Considérons à présent aussi le cas où, outre les deux racines égales $\beta=\alpha$, il se trouve encore une troisième γ qui leur est égale. Or pour les deux premières $\beta=\alpha$ nous venons de réduire leur terme correspondant à cette forme:

$$y^\alpha\mathfrak{A}:\frac{x}{y}+y^\alpha ly\mathfrak{B}:\frac{x}{y},$$

auquel il faut encore ajouter le troisième terme $y^\gamma\mathfrak{C}:\frac{x}{y}$, qui se réuniroit avec le premier. Mais posons à présent $\gamma=a+\omega$, et puisque

$$y^\omega=1+\omega ly+\tfrac{1}{2}\omega^2(ly)^2,$$

il faut aller ici jusqu'au troisième terme, puisque le second se réuniroit avec le second des termes précédens. De là il est clair que ces trois termes, en changeant les fonctions arbitraires, se réduiront aux trois termes suivans:

$$y^\alpha\mathfrak{A}:\frac{x}{y}+y^\alpha ly\mathfrak{B}:\frac{x}{y}+y^\alpha(ly)^2\mathfrak{C}:\frac{x}{y}.$$

Pour en donner un exemple considérons le cas où l'équation pour le nombre n obtient cette forme:

$$1 - 3n + 3nn - n^3 = 0,$$

dont les trois racines sont toutes égales entr'elles, savoir $\alpha = \beta = \gamma = 1$. Ce cas appartient donc à l'équation différentielle du troisième degré

$$Az + BP + CQ + DR = 0,$$

pour laquelle nous avions trouvé:

$$A + nB + n(n-1)C + n(n-1)(n-2)D = 0;$$

ce qui étant développé donne:

$$\left.\begin{array}{l} A + nB + nnC + n^3D \\ \quad - nC - 3nnD \\ \quad + 2nD \end{array}\right\} = 0.$$

Il faudra donc faire:

$$A = 1,\ B - C + 2D = -3,\ C - 3D = +3, \quad \text{et} \quad D = -1,$$

et partant

$$C = 0,\ B = -1, \ \text{et} \ A = 1,$$

de sorte que notre équation différentielle sera:

$$z - P + 0 \cdot Q - R = 0,$$

dont l'intégrale complette, à cause de $\alpha = 1$, sera:

$$y\mathfrak{A} : \frac{x}{y} + yly\mathfrak{B} : \frac{x}{y} + y(ly)^2\mathfrak{C} : \frac{x}{y}.$$

Pour éclaircir ceci par un exemple faisons:

$$\mathfrak{A} = 0, \ \mathfrak{B} = 0 \ \text{et} \ \mathfrak{C} : \frac{x}{y} = \frac{x}{y};$$

de sorte qu'une intégrale particulière sera $x(ly)^2$, d'où nous tirons les différentielles suivantes:

$$\begin{array}{l} \dfrac{\partial z}{\partial x} = (ly)^2, \ \dfrac{\partial z}{\partial y} = \dfrac{2xly}{y}, \\ \dfrac{\partial\partial z}{\partial x^2} = 0, \quad \dfrac{\partial\partial z}{\partial x \partial y} = \dfrac{2ly}{y}, \ \dfrac{\partial\partial z}{\partial y^2} = \dfrac{2x}{yy} - \dfrac{2xly}{yy}, \\ \dfrac{\partial^3 z}{\partial x^3} = 0, \ \dfrac{\partial^3 z}{\partial x^2 \partial y} = 0, \ \dfrac{\partial\partial z}{\partial x \partial y^2} = \dfrac{2}{yy} - \dfrac{2ly}{yy}, \\ \dfrac{\partial^3 z}{\partial y^3} = -\dfrac{4x}{y^3} - \dfrac{2x}{y^3} + \dfrac{4xly}{y^3} = -\dfrac{6x}{y^3} + \dfrac{4xly}{y^3}. \end{array}$$

De là nous tirons

$$z = x(ly)^2, \quad P = x(ly)^2 + 2xly \quad \text{et} \quad R = -2xly,$$

d'où résulte: $z - P - R = 0$, ce qui est parfaitement d'accord.

De là il est déjà très évident, que si le nombre n avoit quatre valeurs égales, savoir $\alpha = \beta = \gamma = \delta$, au lieu des quatre termes qui entrent immédiatement dans l'intégrale, on devra mettre ceux-ci:

$$z = y^\alpha \mathfrak{A}:\frac{x}{y} + y^\alpha ly\,\mathfrak{B}:\frac{x}{y} + y^\alpha (ly)^2 \mathfrak{C}:\frac{x}{y} + y^\alpha (ly)^3 \mathfrak{D}:\frac{x}{y} + \text{etc.}$$

et partant, quel que puisse être le nombre des racines égales, la réduction de l'intégrale n'aura plus aucune difficulté. Au reste on comprend aisément que dans toutes ces formules les deux lettres x et y pourroient être échangées entr' elles.

Pour prouver cela je ferai voir qu'au lieu des termes:

$$y^\alpha \mathfrak{A}:\frac{x}{y} + y^\alpha (ly)\mathfrak{B}:\frac{x}{y}$$

on pourra écrire:

$$x^\alpha \mathfrak{A}:\frac{x}{y} + x^\alpha (lx)\mathfrak{B}:\frac{x}{y}\,.$$

Pour cet effet j'observe que parceque l'un et l'autre terme renferme une fonction arbitraire de $\frac{x}{y}$, on la pourra multiplier par $\frac{x^\alpha}{y^\alpha}$, ce qui donne

$$x^\alpha \mathfrak{A}:\frac{x}{y} + x^\alpha (ly)\mathfrak{B}:\frac{x}{y};$$

ensuite puisque

$$l\frac{x}{y} = lx - ly$$

est aussi fonction de $\frac{x}{y}$, au lieu de $\mathfrak{A}:\frac{x}{y}$ on pourra écrire:

$$\mathfrak{A}:\frac{x}{y} + l\frac{x}{y}\mathfrak{B}:\frac{x}{y},$$

et alors nous aurons

$$x^\alpha \mathfrak{A}:\frac{x}{y} + x^\alpha (lx)\mathfrak{B}:\frac{x}{y};$$

d'où l'on comprend aisément que cette permutation peut toujours avoir lieu.

L'intégration de cette équation différentielle assez générale:

$$Az + BP + CQ + DR + ES + \text{etc.} = 0,$$

où P, Q, R, S etc. marquent les formules différentielles rapportées ci-dessus, pourra être regardée comme un excellent morceau de cette analyse qui traite des fonctions à deux variables, et qu'il faut bien distinguer de l'analyse ordinaire qui ne roule que sur les fonctions à une seule variable. Car il est à présent bien clair que ces deux espèces d'analyse sont très essentiellement différentes entr'elles, non seulement par rapport aux fonctions qui y sont traitées, mais aussi par rapport aux méthodes qu'il y faut employer. C'est pourquoi la dénomination de différences partielles, dont plusieurs Géomètres se servent, pour marquer l'analyse des fonctions à deux variables, ne me paroit pas fort propre pour en exprimer le véritable caractère.

Non obstant cette différence on peut souvent remarquer une belle harmonie entre ces deux espèces d'analyse. Ainsi quand on traite, dans l'analyse ordinaire, cette équation différentielle:

$$Az + Bx\frac{\partial z}{\partial x} + Cx^2\frac{\partial\partial z}{\partial x^2} + Dx^3\frac{\partial^3 z}{\partial x^3} + \text{etc.} = 0;$$

et qu'on demande quelle fonction de x on doit donner à la quantité z, pour que cette équation soit remplie: la méthode ordinaire d'intégrer conduit à cette équation algébrique:

$$A + nB + n(n-1)C + n(n-1)(n-2)D + n(n-1)(n-2)(n-3)E + \text{etc.} = 0;$$

d'où il faut tirer toutes les racines α, β, γ, δ etc. de n, et l'intégrale complette est exprimée de cette manière:

$$z = \mathfrak{A}x^\alpha + \mathfrak{B}x^\beta + \mathfrak{C}x^\gamma + \mathfrak{D}x^\delta + \text{etc.}$$

où les lettres: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc. marquent des constantes arbitraires quelconques. Cette forme a donc un très beau rapport avec la forme de l'intégrale que nous avons trouvée ci-dessus pour la fonction z des deux variables de x et y.

INTEGRATIO AEQUATIONIS DIFFERENTIALIS HUIUS $dy + yydx = \frac{Adx}{(a+2bx+cxx)^2}$

Conventui exhibita die 23. Februarii 1779

Commentatio 734 indicis Enestroemiani

Mémoires de l'académie des sciences de St-Pétersbourg 3 (1809/10), 1811, p. 3—15

1. Ex forma huius aequationis[1]) statim patet, si ea habeat integrale rationale, id necessario hanc speciem habere debere:

$$y = \frac{v}{a+2bx+cxx},$$

cuius formulae differentiale est

$$dy = \frac{dv(a+2bx+cxx) - 2vdx(b+cx)}{(a+2bx+cxx)^2}.$$

Hinc igitur, sublato denominatore, oritur haec aequatio:

$$dv(a+2bx+cxx) - 2vdx(b+cx) + vvdx = Adx.$$

Quaeritur ergo, qualis quantitas pro v accipi debeat, ut isti aequationi satisfiat.

2. Hic iterum facile intelligitur, istum valorem ipsius v aliam formam habere non posse praeter

$$v = f + 2gx + hx^2,$$

1) Vide Commentationes 265, 269: *De aequationibus differentialibus secundi gradus* § 21—27; *De integratione aequationum differentialium* § 67. Novi Comment. acad. sc. Petrop. 7, 1761, p. 178; 8, 1763, p. 39. Vide quoque *Institutionum calculi integralis* vol. II, § 906—910. Leonhardi Euleri *Opera omnia*, series I, vol. 22 et 12. H. D.

et cum hinc sit

$$dv = 2dx(g + hx),$$

facta substitutione ac divisione per dx, resultabit haec aequatio:

$$\left.\begin{array}{l} hhx^4 + 4ghx^3 + 2bhxx + 2ahx + 2ag \\ \qquad\qquad\quad - 2cgxx - 2cfx \;\; - 2bf \\ \qquad\qquad\quad + 2fhxx + 4fgx \;\; + ff \\ \qquad\qquad\quad + 4ggxx \end{array}\right\} = A.$$

Ut iam ista aequatio evadat identica, necesse est, ut singulae potestates ipsius x seorsim se destruant; quare pro potestate quarta tollenda debet esse $h = 0$, hocque modo etiam tertia potestas abscedit, at pro secunda tollenda debet esse $4gg - 2cg = 0$, unde fit $g = \frac{1}{2}c$. Porro si ad nihilum redigantur termini ipsa quantitate x affecti, habebimus $4fg - 2cf = 0$, unde fit $g = \frac{1}{2}c$, quae conditio iam sponte est adimpleta, sicque tantum superest ut reddatur $ff + 2ag - 2bf = A$; quare cum sit $g = \frac{1}{2}c$, statui oportet $ff + ac - 2bf = A$, unde determinatur duplici modo quantitas f, erit enim $f = b \pm \sqrt{bb - ac + A}$.

3. Quo nunc aequatio proposita commodior reddatur, loco $\sqrt{bb - ac + A}$ scribamus k, ut fiat $A = kk - bb + ac$, atque nostra aequatio integranda habebit hanc formam:

$$dy + yydx = \frac{(kk - bb + ac)dx}{(a + 2bx + cxx)^2}$$

et nunc huic aequationi satisfacere vidimus hunc valorem:

$$y = \frac{b \pm k + cx}{a + 2bx + cxx},$$

ita ut iam duos valores simus adepti aequationi nostrae satisfacientes, propter signum ambiguum litterae k assignatum, qui autem non erunt reales nisi k fuerit reale, hoc est nisi fuerit $bb - ac + A$ quantitas positiva. Hic autem probe tenendum est, in his formis neutiquam contineri integrale completum aequationis propositae, propterea quod nulla nova constans arbitraria est introducta, ita ut ista integratio tantum pro particulari sit habenda. Verum aequatio proposita ita est comparata, ut ex quolibet integrali particulari facile integrale completum erui possit, quod quomodo fieri debeat, in aequatione multo generaliori

$$dy + yydx = Vdx$$

ostendisse iuvabit, ubi V denotet functionem quamcunque ipsius x, cuique satisfacere inventus sit hic valor particularis $y = p$, ita ut haec aequatio $dp + ppdx = Vdx$ sit identica, atque nunc ex ipso hoc valore p elici debeat integrale completum.

4. Hunc in finem statuamus integrale completum esse $y = p + z$, factaque substitutione orietur haec aequatio:

$$dp + dz + (pp + 2pz + zz)dx = Vdx,$$

unde si illa aequatio subtrahatur, remanebit ista: $dz + 2pzdx + zzdx = 0$, quae posito $z = \frac{1}{v}$ transformatur in hanc: $dv - 2pvdx = dx$, quae per $e^{-2\int p\,dx}$ multiplicata evadit integrabilis, quippe cuius integrale erit

$$ve^{-2\int p\,dx} = \int e^{-2\int p\,dx}\,dx,$$

quod integrale constantem arbitrariam involvit, ita ut habeamus

$$v = e^{2\int p\,dx}\int e^{-2\int p\,dx}\,dx + Ce^{+2\int p\,dx},$$

quo valore invento erit nostrum integrale completum

$$y = p + \frac{1}{v}.$$

5. Applicemus hanc operationem ad aequationem nostram

$$dy + yydx = \frac{(kk - bb + ac)\,dx}{(a + 2bx + cxx)^2},$$

pro qua invenimus integrale particulare

$$y = p = \frac{b \pm k + cx}{a + 2bx + cxx},$$

ex quo fit

$$2pdx = \frac{2(b \pm k + cx)\,dx}{a + 2bx + cxx},$$

cuius integratio nulla laborat difficultate. Ponamus igitur hoc integrale $\int 2pdx = lq$, ut fiat

$$e^{-2\int p\,dx} = e^{-lq} = \frac{1}{q} \quad \text{et} \quad e^{2\int p\,dx} = q,$$

sicque integrale completum iam erit

$$y = p + \frac{1}{q\int\frac{dx}{q} + Cq}.$$

6. Quoniam vero geminum integrale particulare sumus adepti, propter signum ambiguum quantitatis k, inde integrale completum multo facilius eruitur, id quod etiam in aequatione generali $dy + yydx = Vdx$ ostendamus, cui bina integralia particularia satisfacere assumamus, scilicet primo $y = p$ et secundo $y = q$, ita ut sit

$$\text{tam } dp + ppdx = Vdx$$
$$\text{quam } dq + qqdx = Vdx,$$

subtrahendo ergo utramque ab ipsa aequatione proposita hae duae aequationes orientur:

$$1^0.\ dy - dp + (yy - pp)dx = 0$$

et

$$2^0.\ dy - dq + (yy - qq)dx = 0,$$

unde eliciuntur binae sequentes:

$$\frac{dy - dp}{y - p} + (y + p)dx = 0$$

et

$$\frac{dy - dq}{y - q} + (y + q)dx = 0,$$

quarum haec ab illa subtracta relinquit

$$\frac{dy - dp}{y - p} - \frac{dy - dq}{y - q} + (p - q)dx = 0,$$

cuius integrale manifesto est

$$l\frac{y - p}{y - q} + \int(p - q)dx = C;$$

unde integrale completum iam facile colligitur.

7. Cum enim pro nostra aequatione sit

$$dy + yydx = \frac{(kk - bb + ac)dx}{(a + 2bx + cxx)^2},$$

ubi ex superioribus patet esse

$$p = \frac{b+k+cx}{a+2bx+cxx} \quad \text{et} \quad q = \frac{b-k+cx}{a+2bx+cxx},$$

erit

$$p-q = \frac{2k}{a+2bx+cxx};$$

unde si ponamus

$$\int \frac{2k\,dx}{a+2bx+cxx} = s,$$

habebimus

$$l\frac{y-p}{y-q} + s = C.$$

Hinc colligimus

$$\frac{y-p}{y-q} = \Delta e^{-s},$$

ubi Δ denotat constantem arbitrariam, hincque porro concluditur

$$y = \frac{\Delta q e^{-s} - p}{\Delta e^{-s} - 1} \quad \text{sive} \quad y = \frac{\Delta q - p e^{s}}{\Delta - e^{s}},$$

quod est integrale completum nostrae aequationis.

Quo ista integratio clarior reddatur, eam aliquot exemplis illustremus.

EXEMPLUM I HUIUS AEQUATIONIS

$$dy + yydx = \frac{A\,dx}{(1+xx)^2}$$

8. Hic igitur ante omnia est $a=1$, $b=0$ et $c=1$, hincque erit $A = kk+1$, ideoque $k = \sqrt{(A-1)}$; quam ob rem pro integralibus particularibus habebimus

$$s = 2\sqrt{(A-1)}\int\frac{dx}{1+xx} = 2\sqrt{(A-1)}\ \mathrm{A.tg.}\,x.$$

Porro vero est

$$p = \frac{x+\sqrt{A-1}}{1+xx} \quad \text{et} \quad q = \frac{x-\sqrt{A-1}}{1+xx};$$

unde colligitur integrale completum

$$y = \frac{\Delta(x-\sqrt{A-1}) - e^{s}(x+\sqrt{A-1})}{(1+xx)(\Delta - e^{s})}.$$

9. Quo haec propius ad usum accommodemus, ponamus integrale ita capi debere, ut evanescat posito $x = 0$; hoc autem casu erit $s = 0$, unde constans Δ ita definiri debet, ut fiat

$$0 = \frac{-\Delta\sqrt{A-1}-\sqrt{A-1}}{\Delta-1},$$

unde fit $\Delta = -1$, sicque erit

$$y = \frac{x-\sqrt{A-1}+e^s(x+\sqrt{A-1})}{(1+xx)(1+e^s)},$$

quae expressio semper erit realis, quoties $A - 1$ fuerit quantitas positiva.

10. Cum autem hoc integrale semper debeat esse reale, etiamsi $\sqrt{A-1}$ fuerit imaginarium, ostendendum est, quomodo his casibus imaginaria se mutuo destruant. Quo autem hic calculus facilius expediri possit, ponamus esse $\sqrt{A-1} = \alpha\sqrt{-1}$, tum vero sit brevitatis gratia A. tg. $x = \varphi$, ut sit $x = \text{tg.}\,\varphi$ et $1 + xx = \frac{1}{\cos.\varphi^2}$, sicque nostra aequatio erit

$$y = \frac{(\text{tg.}\varphi-\alpha\sqrt{-1}+e^{2\alpha\varphi\sqrt{-1}}(\text{tg.}\varphi+\alpha\sqrt{-1}))\cos.\varphi^2}{1+e^{2\alpha\varphi\sqrt{-1}}}.$$

11. Quia hic ubique imaginaria occurrunt, atque adeo etiam in exponentibus, ea inde tolli oportet, quod fit ope formulae generalis

$$e^{\omega\sqrt{-1}} = \cos.\omega + \sqrt{-1}\sin.\omega.$$

Nostro casu erit

$$e^{2\alpha\varphi\sqrt{-1}} = \cos.2\alpha\varphi + \sqrt{-1}\sin.2\alpha\varphi,$$

ubi brevitatis gratia loco $2\alpha\varphi$ scribamus tantisper ω. Hoc valore substituto numerator fractionis inventae hanc induet formam:

$$\text{tg.}\,\varphi - \alpha\sqrt{-1} + (\text{tg.}\,\varphi + \alpha\sqrt{-1})(\cos.\,\omega + \sqrt{-1}\sin.\,\omega)$$

sive hanc:

$$\text{tg.}\,\varphi(1 + \cos.\,\omega + \sqrt{-1}\sin.\,\omega) - \alpha\sqrt{-1}(1 - \cos.\,\omega - \sqrt{-1}\sin.\,\omega).$$

Hinc ergo si utrinque multiplicemus per $1 + \cos.\,\omega - \sqrt{-1}\sin.\,\omega$, ut denominator fiat $= 2 + 2\cos.\,\omega = 2(1 + \cos.\,\omega)$, numerator, calculo subducto, evadet $2\,\text{tg.}\,\varphi(1 + \cos.\,\omega) - 2\alpha\sin.\,\omega$, hocque modo tam numerator quam denominator est realis, quocirca integrale nostrum erit

$$y = \frac{\mathrm{tg}.\varphi(1 + \cos.\omega) - \alpha\sin.\omega}{1 + \cos.\omega}\cos.\varphi^2,$$

in quo ergo integrali est

$$\mathrm{tg}.\,\varphi = x, \quad \alpha = -\sqrt{1 - A}, \quad \omega = 2\alpha\varphi = -2\varphi\sqrt{1 - A}.$$

12. Quando igitur in aequatione nostra proposita

$$dy + yydx = \frac{Adx}{(1 + xx)^2}$$

fuerit $A = 1 - \alpha\alpha$, tum posito $x = \mathrm{tg}.\,\varphi$, sumptoque angulo $\omega = 2\alpha\varphi$, erit

$$y = \frac{x(1 + \cos.\omega) - \alpha\sin.\omega}{(1 + xx)(1 + \cos.\omega)},$$

quae expressio adhuc simplicior reddi potest. Cum enim sit

$$\frac{\sin.\omega}{1 + \cos.\omega} = \mathrm{tg}.\tfrac{1}{2}\omega = \mathrm{tg}.\alpha\varphi,$$

erit

$$y = \frac{x - \alpha\,\mathrm{tg}.\alpha\varphi}{1 + xx},$$

qui valor, posito $x = 0$, evanescit.

EXEMPLUM II HUIUS AEQUATIONIS

$$dy + yydx = \frac{Adx}{(1 - xx)^2}$$

13. Hic ergo est $a = 1$, $b = 0$ et $c = -1$; unde fit $A = kk - 1$, ideoque $k = \sqrt{A + 1}$, consequenter

$$s = 2\sqrt{(A + 1)}\int\frac{dx}{1 - xx} = \sqrt{A + 1}\times l\frac{1 + x}{1 - x}.$$

Hinc ergo erit

$$e^s = \left(\frac{1 + x}{1 - x}\right)^k,$$

unde ob

$$p = \frac{k - x}{1 - xx} \quad \text{et} \quad q = \frac{-k - x}{1 - xx}$$

integrale nostrum fiet:

$$y = \frac{\varDelta q - p\left(\frac{1+x}{1-x}\right)^k}{\varDelta - \left(\frac{1+x}{1-x}\right)^k}$$

sive

$$y = \frac{\varDelta (k+x)(1-x)^k + (k-x)(1+x)^k}{(1-xx)((1+x)^k - \varDelta(1-x)^k)}.$$

Quo hanc expressionem ad formam commodiorem redigamus, statuamus $\int \frac{dx}{xx-1} = -\omega$, ut fiat $s = 2k\omega$, atque pro integrali completo nacti sumus

$$y = \frac{\varDelta q - pe^{2k\omega}}{\varDelta - e^{2k\omega}},$$

existente

$$p = \frac{k-x}{1-xx} \quad \text{et} \quad q = \frac{-k-x}{1-xx},$$

ita ut sit

$$y = \frac{\varDelta(k+x) + (k-x)e^{2k\omega}}{(1-xx)(e^{2k\omega} - \varDelta)}.$$

Hic iam loco $\varDelta$ scribamus $\frac{m}{n}$ et supra et infra multiplicemus per $e^{-k\omega}$, eritque

$$y = \frac{me^{-k\omega}(k+x) + n(k-x)e^{k\omega}}{(1-xx)(ne^{k\omega} - me^{-k\omega})},$$

quae forma facilius applicari poterit ad casus, quibus $k = \sqrt{A+1}$ fit quantitas imaginaria, quem casum hic iam omni cura evolvamus.

14. Ponamus igitur formulam $\sqrt{A+1} = k$ esse imaginariam, ita ut sit $k = \alpha\sqrt{-1}$, ideoque $A = -\alpha\alpha - 1$, atque tum habebimus

$$e^{\alpha\omega\sqrt{-1}} = \cos.\alpha\omega + \sqrt{-1}\sin.\alpha\omega \quad \text{et} \quad e^{-\alpha\omega\sqrt{-1}} = \cos.\alpha\omega - \sqrt{-1}\sin.\alpha\omega,$$

quibus valoribus substitutis fiet:

$$y = \frac{+m(\cos.\alpha\omega - \sqrt{-1}\sin.\alpha\omega)(x+\alpha\sqrt{-1}) - n(\cos.\alpha\omega + \sqrt{-1}\sin.\alpha\omega)(x-\alpha\sqrt{-1})}{(1-xx)(n\cos.\alpha\omega + n\sqrt{-1}\sin.\alpha\omega - m\cos.\alpha\omega + m\sqrt{-1}\sin.\alpha\omega)}.$$

15. Hic iam constantes arbitrarias m et n ita assumi convenit, ut saltem denominator evadat realis, quod eveniet ponendo $m = \lambda + \mu\sqrt{-1}$ et $n = -\lambda + \mu\sqrt{-1}$, ita ut fiat $m + n = 2\mu\sqrt{-1}$ et $m - n = 2\lambda$. Hoc

enim modo denominator evadet $-2(1-xx)(\lambda \cos. \alpha\omega + \mu \sin. \alpha\omega)$. Pro numeratore autem evolvendo notetur fore:

$$m(x + \alpha\sqrt{-1}) = \quad \lambda x - \alpha\mu + (\lambda\alpha + \mu x)\sqrt{-1}$$

et

$$n(x - \alpha\sqrt{-1}) = -\lambda x + \alpha\mu + (\lambda\alpha + \mu x)\sqrt{-1}$$

atque ipse numerator erit

$$2 \cos. \alpha\omega(\lambda x - \mu\alpha) + 2 \sin. \alpha\omega(\lambda\alpha + \mu x)$$

hocque modo tota expressio reddita est realis, fit enim:

$$y = \frac{-\cos.\alpha\omega(\lambda x - \mu\alpha) - \sin.\alpha\omega(\lambda\alpha + \mu x)}{(1-xx)(\lambda \cos.\alpha\omega + \mu \sin.\alpha\omega)},$$

quod ergo est integrale completum huius aequationis differentialis:

$$dy + yydx = \frac{-(\alpha\alpha + 1)dx}{(1-xx)^2}.$$

16. Quodsi hanc expressionem ita determinare velimus, ut evanescat casu $x = 0$, quoniam posuimus:

$$\omega = \int \frac{dx}{1-xx} = \tfrac{1}{2} l \frac{1+x}{1-x},$$

hoc casu etiam evadit $\omega = 0$. Sicque esse debebit $0 = \frac{\mu\alpha}{\lambda}$; unde patet statui debere $\mu = 0$; hocque modo integrale desideratum erit:

$$y = \frac{-x \cos.\alpha\omega - \alpha \sin.\alpha\omega}{(1-xx)\cos.\alpha\omega} \quad \text{sive} \quad y = \frac{-x - \alpha\,\mathrm{tg}.\alpha\omega}{1-xx}.$$

Quomodo autem haec expressio satisfaciat, operae pretium erit examinare. Hunc in finem ante omnia notari oportet, ob $d\omega = \frac{dx}{1-xx}$ fore

$$d\,\mathrm{tg}.\alpha\omega = \frac{\alpha dx}{(1-xx)\cos.\alpha\omega^2};$$

tum vero

$$dy = \frac{-dx(1+xx) - \alpha\alpha dx \sec.\alpha\omega^2 - 2\alpha x dx\,\mathrm{tg}.\alpha\omega}{(1-xx)^2}.$$

Quare cum sit

$$yy = \frac{xx + 2\alpha x \operatorname{tg}.\alpha\omega + \alpha\alpha \operatorname{tg}.\alpha\omega^2}{(1-xx)^2},$$

erit

$$\frac{dy}{dx} + yy = \frac{-1-\alpha\alpha}{(1-xx)^2}.$$

INTEGRATIO
GENERALIS AEQUATIONIS PROPOSITAE

17. Quoniam in solutione supra data posuimus $A = kk - bb + ac$, duos casus evolvi oportet, alterum quo $A > ac - bb$, alterum vero quo $A < ac - bb$. Pro priore ergo casu poni poterit $A = kk - bb + ac$, uti supra (§ 3) fecimus, tum vero cum supra (§ 7) posuerimus

$$\int \frac{2k\,dx}{a + 2bx + cxx} = s,$$

nunc statuamus

$$\int \frac{dx}{a + 2bx + cxx} = \omega,$$

ita ut fiat $s = 2k\omega$, atque integrale completum, quod § 13 ita invenimus expressum:

$$y = \frac{\Delta q - pe^{2k\omega}}{\Delta - e^{2k\omega}},$$

nunc, posito $\Delta = \frac{m}{n}$, transformabitur in hanc formam:

$$y = \frac{mqe^{-k\omega} - npe^{+k\omega}}{me^{-k\omega} - ne^{+k\omega}},$$

existente

$$p = \frac{b + k + cx}{a + 2bx + cxx} \quad \text{et} \quad q = \frac{b - k + cx}{a + 2bx + cxx},$$

ubi constans arbitraria continetur in litteris m et n. Hocque modo casu priori est satisfactum, quo est $A = kk - bb + ac$.

18. Aggrediamur nunc alterum casum, quo fit $A < ac - bb$, ac propterea statuamus $A = ac - bb - \alpha\alpha$, qui casus ex praecedente nascitur, ponendo $k = \alpha\sqrt{-1}$. Ante autem vidimus, esse

$$e^{\alpha\omega\sqrt{-1}} = \cos.\alpha\omega + \sqrt{-1}\sin.\alpha\omega \quad \text{et} \quad e^{-\alpha\omega\sqrt{-1}} = \cos.\alpha\omega - \sqrt{-1}\sin.\alpha\omega,$$

unde denominator praecedentis fractionis evadet:

$$m(\cos.\alpha\omega - \sqrt{-1}\sin.\alpha\omega) - n(\cos.\alpha\omega + \sqrt{-1}\sin.\alpha\omega)$$

et iam constantes m et n ita accipiamus, ut iste denominator evadat realis, quod fiet sumendo $m = \lambda + \mu\sqrt{-1}$ et $n = -\lambda + \mu\sqrt{-1}$. Sic enim iste denominator induet hanc formam realem: $2\lambda\cos.\alpha\omega + 2\mu\sin.\alpha\omega$.

19. Pro numeratore autem nunc habebimus

$$mq = \frac{\lambda(b+cx) + \mu\alpha + (\mu(b+cx) - \lambda\alpha)\sqrt{-1}}{a + 2bx + cxx}.$$

Simili modo reperiemus

$$np = \frac{-\lambda(b+cx) - \mu\alpha + (\mu(b+cx) - \lambda\alpha)\sqrt{-1}}{a + 2bx + cxx}.$$

Ponamus autem brevitatis gratia $mq = M + N\sqrt{-1}$ et $np = -M + N\sqrt{-1}$, ita ut sit

$$M = \frac{\lambda(b+cx) + \mu\alpha}{a + 2bx + cxx} \quad \text{et} \quad N = \frac{\mu(b+cx) - \lambda\alpha}{a + 2bx + cxx}.$$

Hocque modo numerator noster erit

$$(\cos.\alpha\omega - \sqrt{-1}\sin.\alpha\omega)(M + N\sqrt{-1}) + (\cos.\alpha\omega + \sqrt{-1}\sin.\alpha\omega)(M - N\sqrt{-1})$$
$$= (2M\cos.\alpha\omega + 2N\sin.\alpha\omega),$$

ita ut nunc etiam numerator habeat formam realem.

20. Cum igitur integrale nostrum completum sit

$$y = \frac{M\cos.\alpha\omega + N\sin.\alpha\omega}{\lambda\cos.\alpha\omega + \mu\sin.\alpha\omega},$$

si loco M et N valores assumtos restituamus, istud integrale evadet:

$$y = \frac{\lambda(b+cx)\cos.\alpha\omega + \mu\alpha\cos.\alpha\omega + \mu(b+cx)\sin.\alpha\omega - \lambda\alpha\sin.\alpha\omega}{(a + 2bx + cxx)(\lambda\cos.\alpha\omega + \mu\sin.\alpha\omega)},$$

ubi ratio inter quantitates λ et μ constantem arbitrariam involvit. Quod si integrale debeat evanescere, sumto $x = 0$, quo casu etiam integrale

$$\omega = \int \frac{dx}{a + 2bx + cxx}$$

evanescet, constantes λ et μ ita determinabuntur, ut fiat

$$0 = \frac{\lambda b + \mu\alpha}{\lambda a} \quad \text{sive} \quad \lambda = \alpha \quad \text{et} \quad \mu = -b,$$

hocque modo integrale nostrum erit

$$y = \frac{\alpha cx \cos.\alpha\omega - \sin.\alpha\omega(\alpha\alpha + bb + bcx)}{(a + 2bx + cxx)(\alpha\cos.\alpha\omega - b\sin.\alpha\omega)}.$$

21. His expeditis geminam integrationem hic sub finem uni obtutui exponamus.

I. Huius aequationis:

$$dy + yydx = \frac{(ac - bb + kk)dx}{(a + 2bx + cxx)^2},$$

integrale completum est:

$$y = \frac{m(b + cx - k)e^{-k\omega} - n(b + cx + k)e^{k\omega}}{(a + 2bx + cxx)(me^{-k\omega} - ne^{k\omega})},$$

ubi litterae m et n arbitrio nostro relinquuntur.

II. Huius aequationis:

$$dy + yydx = \frac{(ac - bb - \alpha\alpha)dx}{(a + 2bx + cxx)^2},$$

integrale completum est:

$$y = \frac{\lambda(b + cx)\cos.\alpha\omega + \mu\alpha\cos.\alpha\omega + \mu(b + cx)\sin.\alpha\omega - \lambda\alpha\sin.\alpha\omega}{(a + 2bx + cxx)(\lambda\cos.\alpha\omega + \mu\sin.\alpha\omega)},$$

ubi litterae λ et μ arbitrio nostro relinquuntur. Pro utroque autem casu ω exprimit integrale formulae

$$\int \frac{dx}{a + 2bx + cxx},$$

quod ita sumi censendum est, ut evanescat posito $x = 0$.

22. Neque vero totum negotium adhuc est confectum, sed unicus adhuc casus evolvendus restat, quo sive $k = 0$, sive $\alpha = 0$, ideoque $A = ac - bb$, quandoquidem hic casus medium interiacet inter binos tractatos, atque ex neutro, non nisi per longas ambages, deduci potest; multo autem magis expediet eum ex primis principiis repetere, ubi bina integralia particularia ita sunt constituta, ut esset

$$p = \frac{b + cx + k}{a + 2bx + cxx} \quad \text{et} \quad q = \frac{b + cx - k}{a + 2bx + cxx},$$

unde fit

$$p - q = \frac{2k}{a + 2bx + cxx},$$

atque pro praesenti casu statui debebit $k = 0$.

23. Spectemus igitur k tanquam quantitatem minimam, ac ponamus brevitatis gratia $p = q + 0$[1]), ut sit

$$0 = \frac{2k}{a + 2bx + cxx},$$

tum vero prima operatio nobis suppeditavit hoc integrale:

$$l\frac{y - p}{y - q} + \int (p - q)\,dx = C,$$

quod igitur nunc erit

$$l\left(1 - \frac{0}{y - q}\right) + \int \frac{2k\,dx}{a + 2bx + cxx} = 2\Delta k,$$

quae ergo expressio, ob

$$\int \frac{dx}{a + 2bx + cxx} = \omega,$$

abit in hanc formam:

$$\frac{-0}{y - q} + 2k\omega = 2\Delta k,$$

ita ut iam sit

$$\frac{0}{y - q} = \frac{2k}{(y - q)(a + 2bx + cxx)} = 2k\omega - 2\Delta k = 2k(\omega - \Delta),$$

1) Hic et in formulis sequentibus 0 non zero, sed quantitatem infinite parvam denotat. H. D.

hincque fit

$$y - q = \frac{1}{(\omega - \varDelta)(a + 2bx + cxx)},$$

consequenter loco q valore substituto prodibit

$$y = \frac{b + cx}{a + 2bx + cxx} + \frac{1}{(\omega - \varDelta)(a + 2bx + cxx)}$$

sive

$$y = \frac{(b + cx)(\omega - \varDelta) + 1}{(\omega - \varDelta)(a + 2bx + cxx)},$$

quod est integrale completum huius casus desiderati, quod ergo neque exponentialia neque circularia involvit.

DE TRANSFORMATIONE FUNCTIONUM DUAS VARIABILES INVOLVENTIUM DUM EARUM LOCO ALIAE BINAE VARIABILES INTRODUCUNTUR

Conventui exhibita die 18. Octobris 1779
Commentatio 737 indicis Enestroemiani
Mémoires de l'académie des sciences de St-Pétersbourg, 3 (1809/10), 1811, p. 43—56

1. Etsi hoc argumentum in tertio volumine Institutionum mearum calculi integralis iam fusius pertractavi, tamen hic methodum sum traditurus, cuius ope tales transformationes multo facilius expediri queant. Si igitur z fuerit functio quaecunque binarum variabilium x et y, harumque loco aliae binae variabiles quaecunque t et u in calculum introducantur, quaestio huc redit, quemadmodum omnes formulae differentiales, ex proposita functione z oriundae, cuiusmodi sunt

$$\left(\frac{\partial z}{\partial x}\right), \left(\frac{\partial z}{\partial y}\right), \left(\frac{\partial \partial z}{\partial x^2}\right), \left(\frac{\partial \partial z}{\partial x\, \partial y}\right), \left(\frac{\partial \partial z}{\partial y^2}\right) \text{ etc.,}$$

per binas novas variabiles t et u exprimantur[1]).

2. Quoniam relatio inter binas variabiles x et y, respectu novarum t et u, cognita assumitur, non solum binae priores x et y tanquam functiones binarum posteriorum t et u spectari poterunt, sed etiam istae t et u erunt certae functiones binarum priorum x et y, quam relationem per sequentes formulas differentiales repraesentabo:

1) Cf. *Institutiones calculi integralis*, vol. III, § 219—239. Leonhardi Euleri *Opera omnia*, series I, vol. 13. H. D.

$$\partial t = P\partial x + Q\,\partial y \quad \text{et} \quad \partial u = R\,\partial x + S\,\partial y,$$

quae ut sint determinatae, necesse est fieri:

$$\left(\frac{\partial P}{\partial y}\right) = \left(\frac{\partial Q}{\partial x}\right) \quad \text{et} \quad \left(\frac{\partial R}{\partial y}\right) = \left(\frac{\partial S}{\partial x}\right),$$

ubi litterae P, Q, R, S non solum tanquam functiones ipsarum x et y sed etiam ipsarum t et u spectari poterunt, ob cognitam rationem, quam hae binae variabiles inter se tenent.

3. His positis primo investigemus valores formularum differentialium primi gradus, quae sunt $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$, quos accipient per novas variabiles t et u. Ac primo quidem, cum sit tam

$$\partial z = \partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right)$$

quam

$$\partial z = \partial t\left(\frac{\partial z}{\partial t}\right) + \partial u\left(\frac{\partial z}{\partial u}\right),$$

loco ∂t et ∂u valores ante stabilitos scribendo prodibit ista aequatio:

$$\partial x\left(\frac{\partial z}{\partial x}\right) + \partial y\left(\frac{\partial z}{\partial y}\right) = P\partial x\left(\frac{\partial z}{\partial t}\right) + Q\partial y\left(\frac{\partial z}{\partial t}\right) + R\partial x\left(\frac{\partial z}{\partial u}\right) + S\partial y\left(\frac{\partial z}{\partial u}\right).$$

Hic evidens est utrinque terminos, eodem differentiali ∂x vel ∂y affectos, seorsim inter se aequari debere; unde colligimus has duas aequationes:

$$\text{I.} \quad \left(\frac{\partial z}{\partial x}\right) = P\left(\frac{\partial z}{\partial t}\right) + R\left(\frac{\partial z}{\partial u}\right),$$

$$\text{II.} \quad \left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right),$$

ubi iam litterae P, Q, R, S tanquam functiones ipsarum t et u spectari poterunt, sicque formulae differentiales $\left(\frac{\partial z}{\partial x}\right)$ et $\left(\frac{\partial z}{\partial y}\right)$ per binas novas $\left(\frac{\partial z}{\partial t}\right)$ et $\left(\frac{\partial z}{\partial u}\right)$ exprimuntur.

4. Multo autem difficilius est hinc valores formularum differentialium secundi gradus, quae sunt $\left(\frac{\partial\partial z}{\partial x^2}\right)$, $\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)$, $\left(\frac{\partial\partial z}{\partial y^2}\right)$ elicere, id quod tamen

sequenti modo satis commode praestari poterit. Incipiamus a prima harum formularum $\left(\frac{\partial\partial z}{\partial x^2}\right)$, quae oritur ex formula $\left(\frac{\partial z}{\partial x}\right)$, si ea differentietur, sumto $\partial y = 0$, et differentiale denuo per ∂x dividatur. At vero, sumto $\partial y = 0$, ex formulis principalibus erit $\partial t = P\partial x$ et $\partial u = R\partial x$, unde fit $\left(\frac{\partial t}{\partial x}\right) = P$ et $\left(\frac{\partial u}{\partial x}\right) = R$. Hinc tantum opus est, ut formulae $P\left(\frac{\partial z}{\partial t}\right) + R\left(\frac{\partial z}{\partial u}\right)$ differentiale per ∂x dividatur, pro casu scilicet $\partial y = 0$. Cum autem iam P et Q sint functiones binarum t et u, earum differentialia talem habebunt formam: $M\partial t + N\partial u$; unde ergo, ob $\left(\frac{\partial t}{\partial x}\right) = P$ et $\left(\frac{\partial u}{\partial x}\right) = R$, pro $\frac{\partial P}{\partial x}$[1]) habebimus $MP + NR$. Simili modo etiam $\frac{\partial Q}{\partial x}$ ad functionem ipsarum t et u reducetur, quae reductio cum per se sit manifesta, in calculo retineamus $\frac{\partial P}{\partial x}$ et $\frac{\partial Q}{\partial x}$. Interim tamen, cum sit $M = \left(\frac{\partial P}{\partial t}\right)$ et $N = \left(\frac{\partial P}{\partial u}\right)$, erit

$$\frac{\partial P}{\partial x} = P\left(\frac{\partial P}{\partial t}\right) + R\left(\frac{\partial P}{\partial u}\right),$$

similique modo erit

$$\frac{\partial Q}{\partial x} = P\left(\frac{\partial Q}{\partial t}\right) + R\left(\frac{\partial Q}{\partial u}\right).$$

5. Superest ergo ut etiam formulas $\left(\frac{\partial z}{\partial t}\right)$ et $\left(\frac{\partial z}{\partial u}\right)$ eadem lege tractemus. Cum igitur in genere sit

$$\partial\cdot\left(\frac{\partial z}{\partial t}\right) = \partial t\left(\frac{\partial\partial z}{\partial t^2}\right) + \partial u\left(\frac{\partial\partial z}{\partial t\,\partial u}\right),$$

hoc per ∂x divisum, ob $\left(\frac{\partial t}{\partial x}\right) = P$ et $\left(\frac{\partial u}{\partial x}\right) = R$, evadet

$$\frac{1}{\partial x}\partial\cdot\left(\frac{\partial z}{\partial t}\right) = P\left(\frac{\partial\partial z}{\partial t^2}\right) + R\left(\frac{\partial\partial z}{\partial t\,\partial u}\right).$$

Simili modo

$$\frac{1}{\partial x}\partial\cdot\left(\frac{\partial z}{\partial u}\right) = P\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + R\left(\frac{\partial\partial z}{\partial u^2}\right).$$

His ergo observatis erit:

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1) In omnibus formulis sequentibus differentiationes ipsorum P, Q, R secundum x et y omissis unciis notantur. H. D.

$$\left(\frac{\partial\partial z}{\partial x^2}\right)=\frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right)+PP\left(\frac{\partial\partial z}{\partial t^2}\right)+PR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right)+RP\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+RR\left(\frac{\partial\partial z}{\partial x^2}\right),$$

quae formula contrahitur in hanc:

$$\left(\frac{\partial\partial z}{\partial x^2}\right)=\frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right)+\frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right)+PP\left(\frac{\partial\partial z}{\partial t^2}\right)+2PR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+RR\left(\frac{\partial\partial z}{\partial u^2}\right).$$

6. Aggrediamur iam secundam formulam $\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)$, quae primo ex formula $\frac{\partial z}{\partial x}$ derivari potest, eam scilicet differentiando, sola y pro variabili sumta, ita ut sit $\partial x=0$. Deinde etiam illa formula derivari potest ex formula $\left(\frac{\partial z}{\partial y}\right)$, eam differentiando, sumta sola x variabili, ideoque $\partial y=0$. Evolvamus primo hoc modo formulam $\left(\frac{\partial z}{\partial x}\right)$, et quia sumto $\partial x=0$ fit $\partial t=Q\,\partial y$ et $\partial u=S\,\partial y$, ideoque $\left(\frac{\partial t}{\partial y}\right)=Q$ et $\left(\frac{\partial u}{\partial y}\right)=S$, hinc ex quantitatibus P et R oriuntur formulae $\frac{\partial P}{\partial y}$ et $\frac{\partial R}{\partial y}$, quarum valores, uti casu praecedente, per se erunt cogniti. Erit scilicet

$$\frac{\partial P}{\partial y}=Q\left(\frac{\partial P}{\partial t}\right)+S\left(\frac{\partial P}{\partial u}\right),$$

similique modo erit

$$\frac{\partial R}{\partial y}=Q\left(\frac{\partial R}{\partial t}\right)+S\left(\frac{\partial R}{\partial u}\right),$$

quarum autem loco retineamus formas $\frac{\partial P}{\partial y}$ et $\frac{\partial R}{\partial y}$. Porro vero habebimus:

$$\frac{1}{\partial y}\partial\cdot\left(\frac{\partial z}{\partial t}\right)=Q\left(\frac{\partial\partial z}{\partial t^2}\right)+S\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$
$$\frac{1}{\partial y}\partial\cdot\left(\frac{\partial z}{\partial u}\right)=Q\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+S\left(\frac{\partial\partial z}{\partial u^2}\right).$$

His igitur colligendis reperiemus:

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)=\frac{\partial P}{\partial y}\left(\frac{\partial z}{\partial t}\right)+PQ\left(\frac{\partial\partial z}{\partial t^2}\right)+PS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\frac{\partial R}{\partial y}\left(\frac{\partial z}{\partial u}\right)+QR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quam etiam hoc modo exprimere licet:

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right)=\frac{\partial P}{\partial y}\left(\frac{\partial z}{\partial t}\right)+\frac{\partial R}{\partial y}\left(\frac{\partial z}{\partial u}\right)+PQ\left(\frac{\partial\partial z}{\partial t^2}\right)+(PS+QR)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+RS\left(\frac{\partial\partial z}{\partial u^2}\right).$$

7. Eundem autem valorem etiam ex altera formula

$$\left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right)$$

elicere licebit, eam differentiando, sumta sola y variabili, ideoque $dy = 0$; unde fit

$$\left(\frac{\partial t}{\partial x}\right) = P \quad \text{et} \quad \left(\frac{\partial u}{\partial x}\right) = R.$$

Hinc igitur primo habemus

$$\frac{\partial Q}{\partial x} = P\left(\frac{\partial Q}{\partial t}\right) + R\left(\frac{\partial Q}{\partial u}\right),$$

similique modo

$$\frac{\partial S}{\partial x} = P\left(\frac{\partial S}{\partial t}\right) + R\left(\frac{\partial S}{\partial u}\right).$$

Deinde erit uti in primo casu:

$$\frac{1}{\partial x}\partial\cdot\left(\frac{\partial z}{\partial t}\right) = P\left(\frac{\partial\partial z}{\partial t^2}\right) + R\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$

et

$$\frac{1}{\partial x}\partial\cdot\left(\frac{\partial z}{\partial u}\right) = P\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + R\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quibus collectis orietur:

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \frac{\partial Q}{\partial x}\left(\frac{\partial z}{\partial t}\right) + PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + QR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \frac{\partial S}{\partial x}\left(\frac{\partial z}{\partial u}\right) + PS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quae etiam hoc modo repraesentari potest:

$$\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \frac{\partial Q}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial x}\left(\frac{\partial z}{\partial u}\right) + PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + (QR + PS)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quae formula cum praecedente egregie convenit; initio enim iam notavimus esse

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \quad \text{et} \quad \frac{\partial R}{\partial y} = \frac{\partial S}{\partial x}.$$

8. Tertia denique formula $\left(\frac{\partial\partial z}{\partial y^2}\right)$ derivari debet ex formula

$$\left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right),$$

sumendo $dx = 0$, unde fit

$$\frac{\partial t}{\partial y} = Q \quad \text{et} \quad \frac{\partial u}{\partial y} = S.$$

Hinc ergo fiet

$$\frac{\partial Q}{\partial y} = Q\left(\frac{\partial Q}{\partial t}\right) + S\left(\frac{\partial Q}{\partial u}\right) \quad \text{et} \quad \frac{\partial S}{\partial y} = Q\left(\frac{\partial S}{\partial t}\right) + S\left(\frac{\partial S}{\partial u}\right).$$

Deinde erit

$$\frac{1}{\partial y}\partial\cdot\frac{\partial z}{\partial t} = Q\left(\frac{\partial\partial z}{\partial t^2}\right) + S\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$

$$\frac{1}{\partial y}\partial\cdot\frac{\partial z}{\partial u} = Q\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + S\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quibus collectis fiet:

$$\left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + QS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + QS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

sive concinnius:

$$\frac{\partial\partial z}{\partial y^2} = \frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + 2QS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right).$$

9. Istos iam valores pro formulis differentialibus secundi gradus inventos heic uni obtutui exponamus:

$$\text{I.}\quad \left(\frac{\partial\partial z}{\partial x^2}\right) = \frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right) + PP\left(\frac{\partial\partial z}{\partial t^2}\right) + 2PR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RR\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\text{II.}\quad \left(\frac{\partial\partial z}{\partial x\,\partial y}\right) = \frac{\partial P}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial y}\left(\frac{\partial z}{\partial u}\right) + PQ\left(\frac{\partial\partial z}{\partial t^2}\right) + (PS + QR)\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

$$\text{III.}\quad \left(\frac{\partial\partial z}{\partial y^2}\right) = \frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + 2QS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right),$$

quibus iungantur formulae differentiales primi gradus:

$$\left(\frac{\partial z}{\partial x}\right) = P\left(\frac{\partial z}{\partial t}\right) + R\left(\frac{\partial z}{\partial u}\right)$$

$$\left(\frac{\partial z}{\partial y}\right) = Q\left(\frac{\partial z}{\partial t}\right) + S\left(\frac{\partial z}{\partial u}\right).$$

Manifestum autem est eadem hac methodo inveniri posse valores formularum differentialium tertii gradus, quae sunt $\frac{\partial^3 z}{\partial t^3}$, $\frac{\partial^3 z}{\partial t^2\,\partial u}$, $\frac{\partial^3 z}{\partial t\,\partial u^2}$, $\frac{\partial^3 z}{\partial u^3}$. Atque adeo ulterius progredi liceret; verum quia formulae nimis complexae essent prodituræ, sufficiat methodum tantum exposuisse.

PROBLEMA

Investigare casus, quibus hanc aequationem differentio-differentialem:

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = vv\left(\frac{\partial\partial z}{\partial y^2}\right)$$

generaliter integrare liceat, ope transformationis ante explicatae.

SOLUTIO

10. Si hic loco formularum $\left(\frac{\partial\partial z}{\partial x^2}\right)$ et $\left(\frac{\partial\partial z}{\partial y^2}\right)$ valores modo inventos substituamus, orietur sequens aequatio:

$$\frac{\partial P}{\partial x}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial R}{\partial x}\left(\frac{\partial z}{\partial u}\right) + PP\left(\frac{\partial\partial z}{\partial t^2}\right) + 2PR\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + RR\left(\frac{\partial\partial z}{\partial u^2}\right)$$
$$= vv\left[\frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + \frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + QQ\left(\frac{\partial\partial z}{\partial t^2}\right) + 2QS\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + SS\left(\frac{\partial\partial z}{\partial u^2}\right)\right].$$

Nunc igitur quaeritur, quomodo novae variabiles t et u accipi oporteat, ut haec aequatio integrationem admittat. Hunc in finem efficiamus primo, ut partes $\left(\frac{\partial\partial z}{\partial t^2}\right)$ se mutuo destruant, quod eveniet si fuerit $PP = QQvv$, ideoque $P = \pm Qv$. Simili modo partes $\left(\frac{\partial\partial z}{\partial u^2}\right)$ se distruent casu $RR = SSvv$, ideoque $R = \pm Sv$. Sumto autem $P = +Qv$, necessario sumi debet $R = -Sv$, quia alioquin etiam partes $\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$ se mutuo destruerent.

11. Sumamus igitur $P = Qv$ et $R = -Sv$, atque nostra aequatio induet hanc formam:

$$\frac{\partial\cdot Qv}{\partial x}\left(\frac{\partial z}{\partial t}\right) - \frac{\partial\cdot Sv}{\partial x}\left(\frac{\partial z}{\partial u}\right) - 2QSvv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)$$
$$= vv\frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + vv\frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 2QSvv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right),$$

sive

$$\frac{\partial\cdot Qv}{\partial x}\left(\frac{\partial z}{\partial t}\right) - \frac{\partial\cdot Sv}{\partial x}\left(\frac{\partial z}{\partial u}\right) = vv\frac{\partial Q}{\partial y}\left(\frac{\partial z}{\partial t}\right) + vv\frac{\partial S}{\partial y}\left(\frac{\partial z}{\partial u}\right) + 4QSvv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right),$$

quae aequatio tribus tantum membris principalibus constat, scilicet:

$$4QSvv\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) + \left(vv\frac{\partial Q}{\partial y} - \frac{\partial\cdot Qv}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right) + \left(vv\frac{\partial S}{\partial y} + \frac{\partial\cdot Sv}{\partial x}\right)\left(\frac{\partial z}{\partial u}\right) = 0.$$

12. Cum igitur sit $P = Qv$ et $R = -Sv$, erit

$$\partial t = Q(v\partial x + \partial y) \quad \text{et} \quad \partial u = S(\partial y - v\partial x);$$

unde patet quantitates Q et S ita accipi debere, ut hae duae formulae integrationem admittant, id quod a valore v potissimum pendet. Quo igitur a simplicissimis incipiamus, sumamus $v = a$, capique poterit tam $Q = 1$ quam $S = 1$, unde fit $P = a$ et $R = -a$; quibus positis aequatio nostra erit

$$4aa\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0, \quad \text{sive} \quad \left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0;$$

qua ergo bis integrata habebimus

$$t = ax + y \quad \text{et} \quad u = y - ax.$$

13. Ad aequationem inventam $\left(\frac{\partial\partial z}{\partial t\,\partial u}\right) = 0$ integrandam statuamus $\left(\frac{\partial z}{\partial u}\right) = s$, fietque $\left(\frac{\partial s}{\partial t}\right) = 0$, ubi sola t variabilis accipitur, existente u constante, quare integrando ponamus $s = \Gamma' : u$, existente $\int \partial u \Gamma' : u = \Gamma : u$. Cum igitur sit $\left(\frac{\partial z}{\partial u}\right) = s = \Gamma' : u$ (ubi iam sola u variabilis sumitur, ita ut t pro constante habeatur), integrando habebimus $z = \Gamma : u + \varDelta : t$, consequenter, loco t et u scriptis eorum valoribus, habebimus huius aequationis:

$$\left(\frac{\partial\partial z}{\partial x^2}\right) = aa\left(\frac{\partial\partial z}{\partial y^2}\right)$$

integrale completum

$$z = \Gamma : (y - ax) + \varDelta : (y + ax),$$

prouti quidem iam dudum constat.

14. Tentemus nunc solutionem generaliorem, sumendo $v = \frac{Y}{X}$, existente X functione ipsius x, et Y functione ipsius y, eritque

$$\partial t = \frac{Q(Y\partial x + X\partial y)}{X} \quad \text{et} \quad \partial u = \frac{S(X\partial y - Y\partial x)}{X},$$

quae ambae formulae integrabiles redduntur, sumendo $Q = \frac{1}{Y}$ et $S = \frac{1}{Y}$; tum enim fit

$$t=\int\frac{\partial x}{X}+\int\frac{\partial y}{Y}\quad\text{et}\quad u=\int\frac{\partial y}{Y}-\int\frac{\partial x}{X};$$

porro vero

$$P=\frac{1}{X}\quad\text{et}\quad R=-\frac{1}{X}.$$

15. His ergo positis aequatio nostra hanc induet formam:

$$\frac{4}{X^2}\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+\left(\frac{X'-Y'}{X^2}\right)\left(\frac{\partial z}{\partial t}\right)-\left(\frac{X'+Y'}{X^2}\right)\left(\frac{\partial z}{\partial u}\right)=0,$$

quae ducta in X^2 reducitur ad hanc:

$$4\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+(X'-Y')\left(\frac{\partial z}{\partial t}\right)-(X'+Y')\left(\frac{\partial z}{\partial u}\right)=0,$$

de qua aequatione observandum est, eam in genere integrabilem esse non posse, nisi alterutra forma $\left(\frac{\partial z}{\partial t}\right)$ vel $\left(\frac{\partial z}{\partial u}\right)$ evanescat. Statuamus igitur $X'-Y'=0$, id quod tantum duplici modo fieri potest: 1^0) scilicet quando vel $X'=0$ et $Y'=0$, hoc est quando utraque functio X et Y est constans, ideoque etiam v constans, quem casum modo ante expedivimus; 2^0) vero quando $X'=b$ et $Y'=b$, unde fit $X=bx$ et $Y=by$, hincque $v=\frac{y}{x}$, ideoque aequatio nostra proposita est

$$\left(\frac{\partial\partial z}{\partial x^2}\right)=\frac{yy}{xx}\left(\frac{\partial\partial z}{\partial y^2}\right),\quad\text{sive}\quad xx\left(\frac{\partial\partial z}{\partial x^2}\right)=yy\left(\frac{\partial\partial z}{\partial y^2}\right),$$

quem ergo casum hic evolvamus.

16. Cum igitur sit $X=bx$ et $Y=by$, erit

$$t=\frac{1}{b}lxy\quad\text{et}\quad u=\frac{1}{b}l\frac{y}{x}.$$

Porro vero erit

$$P=\frac{1}{bx},\quad Q=\frac{1}{by},\quad R=-\frac{1}{bx},\quad S=\frac{1}{by}.$$

Aequatio autem inter t et u erit

$$2\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)-b\left(\frac{\partial z}{\partial u}\right)=0,$$

a cuius ergo integratione tota nostra solutio pendet. Faciamus igitur, ut ante, $\left(\frac{\partial z}{\partial u}\right) = s$, et aequatio nostra erit:

$$2\left(\frac{\partial s}{\partial t}\right) - bs = 0,$$

ubi sola t est variabilis, ideoque u constans; quo notato erit

$$2\partial s - bs\partial t = 0, \quad \text{sive} \quad 2\cdot\frac{\partial s}{s} - b\partial t = 0,$$

cuius integrale est

$$ls - \tfrac{1}{2}bt = \text{Const.} = l\Gamma' : u,$$

et ad numeros transeundo:

$$se^{-\frac{1}{2}bt} = \Gamma' : u, \quad \text{ideoque} \quad s = \left(\frac{\partial z}{\partial u}\right) = e^{\frac{1}{2}bt}\Gamma' : u,$$

sive posito $b = 2c$ erit

$$\left(\frac{\partial z}{\partial u}\right) = e^{ct}\Gamma' : u.$$

In hac autem aequatione iam sola u est variabilis, unde fit $\partial z = e^{ct}\partial u \Gamma' : u$, cuius integrale manifesto est

$$z = e^{ct}\Gamma : u + \varDelta : t.$$

17. Cum igitur sit

$$t = \frac{1}{2c} lxy, \quad \text{ideoque} \quad e^{ct} = \sqrt{xy} \quad \text{et} \quad u = \frac{1}{2c} l\frac{y}{x},$$

hinc erit $\Gamma : u =$ functioni cuicunque $\frac{y}{x}$, similique modo $\varDelta : t =$ functioni cuicunque ipsius xy, consequenter integrale nostrum completum erit

$$z = \sqrt{xy}\,\Sigma : \frac{y}{x} + \Theta : xy;$$

ubi prius membrum multiplicari potest per $\sqrt{\frac{y}{x}}$, quo facto erit

$$z = y\,\Sigma : \frac{y}{x} + \Theta : xy.$$

Ubi observasse iuvabit, cum $\Sigma : \frac{y}{x}$ comprehendat omnes functiones nullius dimensionis ipsarum x et y, prius membrum denotare omnes functiones ipsarum x et y unius dimensionis. Praeter hos autem duos casus modo tractatos haud patet alios exhiberi posse, quibus aequationem in problemate propositam complete integrare liceat.

PROBLEMA

Investigare casus, quibus haec aequatio differentialis secundi gradus:

$$xx\left(\frac{\partial\partial z}{\partial x^2}\right) - fxy\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) + gyy\left(\frac{\partial\partial z}{\partial y^2}\right) = 0$$

ope transformationis hic traditae complete integrari possit.

SOLUTIO

18. Loco binarum variabilium x et y, quarum functio est z, introducantur binae aliae t et u, quarum ad illas relatio his aequationibus exprimatur:

$$\partial t = P\partial x + Q\,\partial y \quad \text{et} \quad \partial u = R\,\partial x + S\,\partial y,$$

atque ex formulis supra datis colligamus primo coefficientem termini $\left(\frac{\partial\partial z}{\partial t^2}\right)$, qui est

$$PPxx - fPQxy + gQQyy.$$

Similique modo coefficiens termini $\left(\frac{\partial\partial z}{\partial u^2}\right)$ erit

$$RRxx - fRSxy + gSSyy,$$

quos ambos evanescentes reddamus, quod quo commodius fieri queat, statuamus $f = a + b$ et $g = ab$; hocque modo prior coefficiens resolvitur in hos factores: $(Px - aQy)\,(Px - bQy)$, qui ut evanescat ponamus $Px = aQy$. Alter vero coefficiens in hos factores resolvitur: $(Rx - aSy)\,(Rx - bSy)$, qui ut evanescat faciamus $Rx = bSy$.

19. Cum igitur sit

$$P = \frac{aQy}{x} \quad \text{et} \quad R = \frac{bSy}{x},$$

formulae principales pro ∂t et ∂u erunt:

$$\partial t = Q\left(\frac{ay\partial x + x\partial y}{x}\right) \quad \text{et} \quad \partial u = S\left(\frac{by\partial x + x\partial y}{x}\right),$$

quae ambae fient integrabiles sumendo

$$Q = \frac{1}{y} \quad \text{et} \quad S = \frac{1}{y}.$$

Sic enim fiet

$$\partial t = a\cdot\frac{\partial x}{x} + \frac{\partial y}{y} \quad \text{et} \quad \partial u = b\cdot\frac{\partial x}{x} + \frac{\partial y}{y};$$

unde integralia erunt

$$t = alx + ly \quad \text{et} \quad u = blx + ly,$$

sive

$$t = lx^a y \quad \text{et} \quad u = lx^b y.$$

Sumtis autem $Q = \frac{1}{y}$ et $S = \frac{1}{y}$, erit $P = \frac{a}{x}$ et $R = \frac{b}{x}$.

20. His iam valoribus substitutis, quia termini $\left(\frac{\partial\partial z}{\partial t^2}\right)$ et $\left(\frac{\partial\partial z}{\partial u^2}\right)$ iam ad nihilum sunt perducti, coefficiens termini $\left(\frac{\partial\partial z}{\partial t\partial u}\right)$ reperietur

$$2PRxx - f(QR + PS)xy + 2gQSyy,$$

qui ob $f = a + b$ et $g = ab$, facta substitutione litterarum maiuscularum, fit $4ab - (a + b)^2 = -(a - b)^2$, ita ut hic terminus iam sit

$$-(a - b)^2\left(\frac{\partial\partial z}{\partial t\partial u}\right).$$

Porro vero termini $\left(\frac{\partial z}{\partial t}\right)$ coefficiens erit

$$xx\frac{\partial P}{\partial x} - fxy\frac{\partial Q}{\partial x} + gyy\frac{\partial Q}{\partial y},$$

qui ob

$$\frac{\partial P}{\partial x} = \frac{-a}{xx}, \quad \frac{\partial Q}{\partial x} = 0 \quad \text{et} \quad \frac{\partial Q}{\partial y} = \frac{-1}{yy}$$

abit in hanc formam: $-a(b+1)$. Similique modo coefficiens termini $\left(\frac{\partial z}{\partial u}\right)$ colligitur fore

$$xx\frac{\partial R}{\partial x}-fxy\frac{\partial S}{\partial x}+gyy\frac{\partial S}{\partial y}=-b(a+1).$$

21. Aequatio igitur resolvenda nunc hanc induet formam:

$$(a-b)^2\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)+a(b+1)\left(\frac{\partial z}{\partial t}\right)+b(a+1)\left(\frac{\partial z}{\partial u}\right)=0,$$

de qua autem ante omnia notari oportet, eam nullo modo adhuc cognito tractari posse, nisi alteruter posteriorum terminorum evanescat. Statuamus igitur $b=-1$; unde fit

$$f=a-1,\quad g=-a\quad \text{et}\quad t=lx^a y\quad \text{et}\quad u=l\frac{y}{x}.$$

Aequatio vero resolvenda erit

$$(a+1)^2\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)-(a+1)\left(\frac{\partial z}{\partial u}\right)=0.$$

Hinc ergo si ponamus $\left(\frac{\partial z}{\partial u}\right)=v$, ob $\left(\frac{\partial\partial z}{\partial t\,\partial u}\right)=\left(\frac{\partial v}{\partial t}\right)$, erit

$$(a+1)^2\left(\frac{\partial v}{\partial t}\right)=(a+1)v,\quad \text{sive}\quad (a+1)\left(\frac{\partial v}{\partial t}\right)=v,$$

ubi littera u tanquam constans est spectanda, quo observato erit $(a+1)\partial v=v\partial t$, ideoque $\frac{\partial v}{v}=\frac{\partial t}{a+1}$, unde fit $lv=\frac{t}{a+1}+l\Gamma':u$, sicque numeris sumtis erit

$$v=e^{\frac{t}{a+1}}\Gamma':u.$$

22. Cum igitur posuerimus $v=\left(\frac{\partial z}{\partial u}\right)$, ita ut nunc t pro constante sit habenda, erit

$$\left(\frac{\partial z}{\partial u}\right)=e^{\frac{t}{a+1}}\Gamma':u,\quad \text{sive}\quad \partial z=e^{\frac{t}{a+1}}\partial u\Gamma':u,$$

unde ob $\int\partial u\Gamma':u=\Gamma:u$ habebimus

$$z=e^{\frac{t}{a+1}}\Gamma:u+\Delta:t,$$

quae expressio, ob binas functiones arbitrarias, utique praebet integrale completum aequationis propositae, casu scilicet quo $f=a-1$ et $g=-a$.

23. Quo nunc hanc formam ad variabiles x et y transferamus, notemus primo esse $t = lx^a y$, unde fit $e^t = x^a y$, hincque

$$e^{\frac{t}{a+1}} = x^{\frac{a}{a+1}} y^{\frac{1}{a+1}} = \sqrt[a+1]{x^a y}\;;$$

tum vero functio quaecunque ipsius t erit etiam functio quaecunque ipsius $x^a y$, unde pro $\varDelta : t$ scribere licebit $\varDelta : x^a y$. Deinde cum sit $u = l\frac{y}{x}$, eius functio quaecunque etiam erit functio ipsius $\frac{y}{x}$, sicque loco $\varGamma : u$ nunc habebimus $\varGamma : \frac{y}{x}$. Hinc huius aequationis differentio-differentialis:

$$xx\left(\frac{\partial\partial z}{\partial x^2}\right) - (a-1)xy\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) - ayy\left(\frac{\partial\partial z}{\partial y^2}\right) = 0,$$

integrale completum erit:

$$z = \sqrt[a+1]{x^a y}\cdot\varGamma : \frac{y}{x} + \varDelta : x^a y\,.$$

Quoniam igitur ista aequatio abit in eam quam praecedente problemate invenimus casu $a = 1$, eadem forma integralis prodibit, quam supra (§ 17) invenimus, scilicet:

$$z = \sqrt{xy}\varGamma : \frac{y}{x} + \varDelta : xy\,.$$

24. Prius membrum illius formae integralis multo simplicius exprimi potest, dum scilicet eius factor prior per quandam functionem ipsius $\frac{y}{x}$ multiplicetur vel dividatur. Dividatur ergo per $\sqrt[a+1]{\frac{y}{x}}$, prodibit

$$z = x\varGamma : \frac{y}{x} + \varDelta : x^a y;$$

ubi notandum est prius membrum $x\varGamma : \frac{y}{x}$ continere omnes functiones homogeneas unius dimensionis ipsarum x et y. Observetur hic, si etiam fuerit $a = -1$, ita ut aequatio proposita sit

$$xx\left(\frac{\partial\partial z}{\partial x^2}\right) + 2xy\left(\frac{\partial\partial z}{\partial x\,\partial y}\right) + yy\left(\frac{\partial\partial z}{\partial y^2}\right) = 0,$$

tum integrale completum fore $z = x\varGamma : \frac{y}{x} + \varDelta : \frac{y}{x}$. Ubi notandum, etiamsi duae functiones eiusdem formae $\frac{y}{x}$ occurrant, eas in unam contrahi non posse, propterea quod prior multiplicata est per x.

INTEGRATIO GENERALIS AEQUATIONUM DIFFERENTIALIUM LINEARIUM CUIUSCUNQUE GRADUS ET QUOTCUNQUE VARIABILES INVOLVENTIUM

Conventui exhibita die 28. Octobris 1779
Commentatio 741 indicis ENESTROEMIANI
Mémoires de l'académie des sciences de Saint Pétersbourg 4 (1811), 1813, p. 43—51

1. Si fuerit functio quotcunque variabilium z, y, x, u etc. determinanda ex aequatione differentiali cuiuscunque gradus, in cuius singulis terminis quantitas V, cum suis differentialibus, unicam obtineat dimensionem, atque insuper coefficientes singulorum terminorum fuerint constantes, tales aequationes hic voco lineares, et quemadmodum eas integrari oporteat, investigabo[1]).

2. Forma completa talium aequationum primo continebit ipsam quantitatem quaesitam V. Deinde occurrent differentialia primi gradus, quae sunt $\left(\frac{\partial V}{\partial z}\right)$, $\left(\frac{\partial V}{\partial y}\right)$, $\left(\frac{\partial V}{\partial x}\right)$ etc., quorum ergo numerus est $= n$, siquidem n fuerit numerus variabilium z, y, x, u etc. Sequuntur termini differentiales secundi gradus: $\left(\frac{\partial\partial V}{\partial z^2}\right)$, $\left(\frac{\partial\partial V}{\partial z\,\partial y}\right)$, $\left(\frac{\partial\partial V}{\partial y^2}\right)$, $\left(\frac{\partial\partial V}{\partial z\,\partial x}\right)$ etc., quorum numerus est $\frac{n(n+1)}{1\cdot 2}$. Terminorum porro differentialium tertii gradus: $\left(\frac{\partial^3 V}{\partial z^3}\right)$, $\left(\frac{\partial^3 V}{\partial z^2\,\partial y}\right)$, $\left(\frac{\partial^3 V}{\partial z\,\partial y^2}\right)$,

1) Cf. *Institutiones calculi integralis*, vol. III, § 395, 403—404, 411—429, 495—510. Vide notam 2, p. 44 huius voluminis. H. D.

$\left(\frac{\partial^3 V}{\partial y^3}\right)$, $\left(\frac{\partial^3 V}{\partial y^2 \partial x}\right)$ etc. numerus est $\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}$, sicque porro. Atque hi singuli termini, per quantitates constantes multiplicati, exhibebunt generatim aequationem differentialem linearem cuiuscunque gradus, cuius integrationem hic accuratius sum perscrutaturus.

3. Tales ergo aequationes omnes in hac forma generali continebuntur:

$$\begin{aligned} 0 = AV &+ B\left(\frac{\partial V}{\partial z}\right) + C\left(\frac{\partial V}{\partial y}\right) + D\left(\frac{\partial V}{\partial x}\right) + \text{etc.} \\ &+ E\left(\frac{\partial\partial V}{\partial z^2}\right) + F\left(\frac{\partial\partial V}{\partial y^2}\right) + G\left(\frac{\partial\partial V}{\partial x^2}\right) + H\left(\frac{\partial\partial V}{\partial z\partial y}\right) + I\left(\frac{\partial\partial V}{\partial z\partial x}\right) + K\left(\frac{\partial\partial V}{\partial x\partial y}\right) + \text{etc.} \\ &+ L\left(\frac{\partial^3 V}{\partial z^3}\right) + M\left(\frac{\partial^3 V}{\partial y^3}\right) + \text{etc.} \\ &\qquad\text{etc.} \end{aligned}$$

Huic autem formae satis constat satisfacere talem formam integralem: $V = e^{\alpha z+\beta y+\gamma x+\text{etc.}}$. Hinc enim erit:

$$\begin{aligned} &\left(\frac{\partial V}{\partial z}\right) = \alpha e \cdots\cdots, && \left(\frac{\partial V}{\partial y}\right) = \beta e \cdots\cdots, \\ &\left(\frac{\partial V}{\partial x}\right) = \gamma e \cdots\cdots, && \left(\frac{\partial\partial V}{\partial z^2}\right) = \alpha\alpha e \cdots\cdots, \\ &\left(\frac{\partial\partial V}{\partial y^2}\right) = \beta\beta e \cdots\cdots, && \left(\frac{\partial\partial V}{\partial x^2}\right) = \gamma\gamma e \cdots\cdots, \\ &\left(\frac{\partial\partial V}{\partial z\partial y}\right) = \alpha\beta e \cdots\cdots, && \left(\frac{\partial\partial V}{\partial z\partial x}\right) = \alpha\gamma e \cdots\cdots, \\ &\left(\frac{\partial\partial V}{\partial x\partial y}\right) = \beta\gamma e \cdots\cdots, && \left(\frac{\partial^3 V}{\partial z^3}\right) = \alpha^3 e \cdots\cdots, \\ &\left(\frac{\partial^3 V}{\partial y^3}\right) = \beta^3 e \cdots\cdots, && \left(\frac{\partial^3 V}{\partial x^3}\right) = \gamma^3 e \cdots\cdots \end{aligned}$$

et ita porro; quibus substitutis, quia totam aequationem per $e^{\alpha z+\beta y+\gamma x+\text{etc.}}$ dividere licet, patet, litteras assumtas α, β, γ etc. hac aequatione determinari debere:

$$\begin{aligned} 0 = A &+ B\alpha + C\beta + D\gamma + \text{etc.} \\ &+ E\alpha^2 + F\beta^2 + G\gamma^2 + H\alpha\beta + I\alpha\gamma + K\beta\gamma + \text{etc.} \\ &+ L\alpha^3 + M\beta^3 + N\gamma^3 + \text{etc.} \\ &\qquad\text{etc.} \end{aligned}$$

4. Ex hac igitur aequatione, quam aequationis differentialis propositae *vicariam* appellare liceat, litterarum assumtarum α, β, γ etc. quaelibet per reliquas, scilicet α per β, γ etc. definiri poterit, idque tot modis, quoti gradus differentialis fuerit aequatio, ita ut hic reliquae litterae β, γ etc. prorsus ab arbitrio nostro pendeant. Sic igitur singuli valores his litteris tributi praebebunt formulam determinatam $e^{\alpha z+\beta y+\gamma x+\text{ etc.}}$, cuiusmodi ergo formularum numerus prorsus erit infinitus, atque adeo eo altioris gradus, quo maior fuerit numerus litterarum β, γ etc., per quas primam α determinaverimus. Quod si ergo singulae istae formulae per coefficientes constantes arbitrarios multiplicentur et in unam summam aggregentur, habebitur expressio maxime generalis, valorem quaesitum V exhibens, quam igitur tanquam integrale completum spectare licebit.

5. Verum talis expressio, ob terminorum numerum infinities infinitum, hoc laborat incommodo, quod inde verus integralis valor perspici nequeat, quapropter porro erit investigandum, utrum talis expressio, in infinitum excurrens, non per certam formulam, seu functionem finitam, repraesentari possit, hocque commode succedet, quoties relationem inter litteras α, β, γ etc. tali simplici formula $a\alpha + b\beta + c\gamma + \text{etc.} \cdots + k = 0$ exhibere licet, id quod evenit, quando ista formula fuerit factor formae vicariae supra allatae (§ 3); tum enim omnes illas formulas exponentiales, numero infinitas, per certas functiones repraesentare licebit, quemadmodum in sequentibus ostendemus.

6. Cum forma vicaria ex ipsa aequatione proposita sit formata, evidens est quemadmodum vicissim ipsa aequatio ex forma vicaria derivari possit. Si enim formula $k + a\alpha + b\beta + c\gamma + \text{etc.}$ fuerit factor formae vicariae, illi respondebit haec aequatio differentialis primi gradus:

$$kV + a\left(\frac{\partial V}{\partial z}\right) + b\left(\frac{\partial V}{\partial y}\right) + c\left(\frac{\partial V}{\partial x}\right) + \text{etc.} = 0,$$

cuius ergo integrale simul erit quoque integrale ipsius aequationis propositae[1]). Ad hoc igitur investigandum statuamus in genere esse

$$\partial V = p\,\partial z + q\,\partial y + r\,\partial x + \text{etc.},$$

1) Cf. Commentationem 62: *De integratione aequationum differentialium altiorum graduum.* Miscellanea Berol. 7, (1743), p. 205. LEONHARDI EULERI *Opera omnia, series* I, vol. 22. H. D.

ut ista aequatio in hanc abeat:

$$kV + ap + bq + cr + \text{etc.} = 0.$$

7. Ex hac iam aequatione quaeramus valorem ipsius

$$p = -\frac{kV}{a} - \frac{bq}{a} - \frac{cr}{a} - \text{etc.},$$

qui in illa formula assumta substitutus dabit:

$$\partial V = -\frac{kV\partial z}{a} + q\left(\partial y - \frac{b\,\partial z}{a}\right) + r\left(\partial x - \frac{c\,\partial z}{a}\right)\ ^{1)},$$

quae aequatio hoc modo repraesentetur:

$$\partial V + \frac{kV\partial z}{a} = q\left(\partial y - \frac{b\,\partial z}{a}\right) + r\left(\partial x - \frac{c\,\partial z}{a}\right),$$

cuius membrum sinistrum integrabile manifesto redditur, si ducatur in $e^{\frac{kz}{a}}$, siquidem eius integrale erit $e^{\frac{kz}{a}}V$, sicque nostra aequatio ita se habebit:

$$\partial \cdot e^{\frac{kz}{a}}V = e^{\frac{kz}{a}} \times q\left(\partial y - \frac{b\,\partial z}{a}\right) + e^{\frac{kz}{a}} \times r\left(\partial x - \frac{c\,\partial z}{a}\right).$$

Haud difficulter autem intelligitur, quantitates q et r semper ita accipi posse, ut membrum dextrum etiam integrationem admittat.

8. Quod quo facilius appareat, statuamus

$$y - \frac{bz}{a} = s \quad \text{et} \quad x - \frac{cz}{a} = t,$$

fietque nostra aequatio:

$$\partial \cdot e^{\frac{kz}{a}} \times V = e^{\frac{kz}{a}} \times (q\partial s + r\partial t),$$

ubi membrum postremum in genere refert differentiale functionis cuiuscunque binarum variabilium s et t, unde colligitur integrando formulam $e^{\frac{kz}{a}}V$ aequari functioni cuicunque binarum variabilium s et t, quam more iam recepto hoc modo repraesentemus: $\Gamma: \overline{s,t}$; ergo loco s et t restitutis valoribus orietur iste valor:

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1) In paragraphis 7—10 formulae saepe casum trium variabilium spectant. Correctionem ob observationes ultimae paragraphi omittere maluimus. H. D.

$$V = e^{-\frac{kz}{a}} \Gamma : \overline{\left(y - \frac{bz}{a}\right), \left(x - \frac{cz}{a}\right)}.$$

Hic scilicet valor aequationi propositae convenit, quoties eius forma vicaria factorem habuerit $k + a\alpha + b\beta + c\gamma +$ etc.

9. Quodsi forma vicaria praeterea alium habeat factorem simplicem, qui sit $k' + a'\alpha + b'\beta + c'\gamma +$ etc., ex eo simili modo deducetur valor pro littera V, qui erit

$$V = e^{-\frac{k'z}{a'}} \varDelta : \overline{\left(y - \frac{b'z}{a'}\right), \left(x - \frac{c'z}{a'}\right)},$$

qui cum praecedente quomodocunque coniungi poterit. Atque si forma vicaria in meros factores simplices, numero n, resolvi se patiatur, qui sint

$$k + a\alpha + b\beta + c\gamma + \text{etc.},\quad k' + a'\alpha + b'\beta + c'\gamma + \text{etc.},\quad k'' + a''\alpha + b''\beta + c''\gamma + \text{etc.},\ \text{etc.},$$

tum adeo integrale completum quantitatis V assignare poterimus, quod erit:

$$V = e^{-\frac{kz}{a}} \Gamma : \overline{\left(y - \frac{bz}{a}\right), \left(x - \frac{cz}{a}\right)} + e^{-\frac{k'z}{a'}} \varDelta : \overline{\left(y - \frac{b'z}{a'}\right), \left(x - \frac{c'z}{a'}\right)} + e^{-\frac{k''z}{a''}} \Sigma : \overline{\left(y - \frac{b''z}{a''}\right), \left(x - \frac{c''z}{a''}\right)} + \text{etc.},$$

ubi characteres Γ, $\varDelta$, Σ etc. denotant functiones quascunque arbitrarias quantitatum subnexarum.

10. Verum eadem integralia, per functiones expressa, ex formula initio assumta $V = e^{\alpha z + \beta y + \gamma x}$ derivari possunt. Cum enim factor formae nostrae vicariae:

$$k + a\alpha + b\beta + c\gamma + \text{etc.} = 0 \quad \text{praebeat} \quad \alpha = -\frac{k}{a} - \frac{b\beta}{a} - \frac{c\gamma}{a} - \text{etc.},$$

exponens ipsius e erit

$$-\frac{kz}{a} + \beta\left(y - \frac{bz}{a}\right) + \gamma\left(x - \frac{cz}{a}\right),$$

qui, posito ut ante brevitatis gratia

$$y - \frac{bz}{a} = s \quad \text{et} \quad x - \frac{cz}{a} = t,$$

erit

$$-\frac{kz}{a} + \beta s + \gamma t,$$

sicque habebitur:

$$V = e^{-\frac{kz}{a}+\beta s+\gamma t} = e^{-\frac{kz}{a}} \cdot e^{\beta s} \cdot e^{\gamma t};$$

ubi probe notandum est litteras β et γ omnes possibiles valores recipere, ita ut $e^{\beta s}$ complectatur summam omnium eius valorum, qui ex littera β oriri possunt, atque adeo omnes isti valores per quantitates constantes quascunque multiplicati sunt intelligendi. Aggregatum igitur omnium horum infinitorum valorum per $\int e^{\beta s}$ designemus. Similique modo $\int e^{\gamma t}$ omnes valores possibiles, qui ex variatione litterae γ nasci possunt, complectatur.

11. Iam haud difficulter intelligitur, istam formam $\int e^{\beta s}$ omnes plane functiones ipsius s exhibere posse. Quod quo clarius appareat, ponamus $s = lp$, fietque $e^{\beta s} = p^\beta$ et $\int e^{\beta s} = \int p^\beta$. Notum autem est, omnes functiones ipsius p resolvi posse in series, quarum termini procedant secundum potestates ipsius p, sicque formula $\int p^\beta$ complectitur omnes valores possibiles ipsius p, ideoque etiam omnes functiones ipsius s, quas ergo hoc charactere $\Gamma : s$ repraesentare licet. Simili modo, posito $t = lq$, patet $\int e^{\gamma t}$ aequivalere huic functioni: $\varDelta : t$.

12. Praeterea, quia singulas partes utriusque functionis in se multiplicare licet, manifestum est formulam $e^{\beta s+\gamma t}$ [1]) non solum productum functionis ipsius s et functionis ipsius t indicare, sed etiam omnes plane functiones utcunque ex binis quantitatibus s et t formatas involvere, quam ergo expressionem hoc charactere: $\Gamma : \overline{s, t}$ repraesentamus. Hinc igitur, cum sit

$$s = y - \frac{bz}{a} \quad \text{et} \quad t = x - \frac{cz}{a},$$

erit, uti ante per integrationem collegimus:

$$V = e^{-\frac{kz}{a}} \Gamma : \overline{\left(y - \frac{bz}{a}\right), \ \left(x - \frac{cz}{a}\right)}.$$

13. Hoc autem modo per functiones integralia huiusmodi aequationum differentialium exprimere non licet, nisi quatenus earum forma vicaria factores simplices comprehendit. Nisi enim hoc eveniat, integralia aliter exhiberi nequeunt, nisi omnia integralia particularia, quae ex formula $e^{\alpha z+\beta y+\gamma x+\text{etc.}}$

1) Editio princeps: $e^{\beta s+\gamma t}+1$. Correxit H. D.

oriuntur, in unam summam colligendo. Quod quo clarius appareat, consideremus istum casum specialem:

$$\left(\frac{\partial\partial V}{\partial z^2}\right) = \left(\frac{\partial\partial V}{\partial x\,\partial y}\right),$$

cuius forma vicaria est $\alpha\alpha = \beta\gamma$, quae certe in factores simplices nullo modo resolvi potest. Hinc autem fit $\alpha = \sqrt{\beta\gamma}$, ideoque

$$V = e^{z\sqrt{\beta\gamma}+\beta y+\gamma z},$$

ubi binis litteris β et γ omnes possibiles valores tribui sunt censendae, quas autem nullo modo sub quapiam functione definita complecti licet. Quo hoc clarius appareat ponamus $z = lp$, $y = lq$ et $x = lr$, ut fiat

$$V = p^{\sqrt{\beta\gamma}}\, q^{\beta} r^{\gamma}.$$

Quod si iam hic litteris α, β, γ tantum valores integros tribuantur, prodibit talis aequatio:

$$V = \mathfrak{A}pqr + \mathfrak{B}p^{\sqrt{2}}q^2r + \mathfrak{C}p^{\sqrt{3}}q^3r + \text{etc.},$$

quos diversos terminos sub nulla certa functione complecti licet.

14. Interim tamen, si aequatio vicaria resolutionem in duos factores, etsi non simplices, admittat, tum integrale aequationis propositae ad integrationem duarum aequationum inferioris gradus reducitur. Quodsi enim aequatio differentialis proposita ascendat ad gradum differentialium $m = \mu + \nu$, eiusque forma vicaria habeat duos factores, alterum gradus μ, alterum vero gradus ν, si ex his vicissim formentur aequationes differentiales, altera ad gradum μ, altera ad gradum ν assurget. Ponamus ergo integrationem prioris praebere $V = P$, posterioris vero $V = Q$, atque manifestum est, ipsius aequationis propositae integrale fore $AP + BQ$; atque adeo, si illa integralia fuerint completa, etiam hoc erit completum[1]).

15. Quae hactenus sunt tradita, proprie quidem ad functiones trium variabilium z, y, x sunt accomodata: facile autem intelligitur, eadem praecepta pariter locum obtinere tam pro paucioribus, quam pro pluribus variabilibus.

1) Quod in peculiaribus casibus falsum esse potest. H. D.

sicque habebitur:

$$V = e^{-\frac{kz}{a}+\beta s+\gamma t} = e^{-\frac{kz}{a}}\cdot e^{\beta s}\cdot e^{\gamma t};$$

ubi probe notandum est litteras β et γ omnes possibiles valores recipere, ita ut $e^{\beta s}$ complectatur summam omnium eius valorum, qui ex littera β oriri possunt, atque adeo omnes isti valores per quantitates constantes quascunque multiplicati sunt intelligendi. Aggregatum igitur omnium horum infinitorum valorum per $\int e^{\beta s}$ designemus. Similique modo $\int e^{\gamma t}$ omnes valores possibiles, qui ex variatione litterae γ nasci possunt, complectatur.

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12. Praeterea, quia singulas partes utriusque functionis in se multiplicare licet, manifestum est formulam $e^{\beta s+\gamma t}$ 1) non solum productum functionis ipsius s et functionis ipsius t indicare, sed etiam omnes plane functiones utcunque ex binis quantitatibus s et t formatas involvere, quam ergo expressionem hoc charactere: $\Gamma : \overline{s, t}$ repraesentamus. Hinc igitur, cum sit

$$s = y - \frac{bz}{a} \quad \text{et} \quad t = x - \frac{cz}{a},$$

erit, uti ante per integrationem collegimus:

$$V = e^{-\frac{kz}{a}}\Gamma : \overline{\left(y - \frac{bz}{a}\right),\ \left(x - \frac{cz}{a}\right)}.$$

13. Hoc autem modo per functiones integralia huiusmodi aequationum differentialium exprimere non licet, nisi quatenus earum forma vicaria factores simplices comprehendit. Nisi enim hoc eveniat, integralia aliter exhiberi nequeunt, nisi omnia integralia particularia, quae ex formula $e^{\alpha z+\beta v+\gamma x+\text{etc.}}$

1) Editio princeps: $e^{\beta s+\gamma t} + 1$. Correxit H. D.

oriuntur, in unam summam colligendo. Quod quo clarius appareat, consideremus istum casum specialem:

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$$V = e^{z\sqrt{\beta\gamma}+\beta y+\gamma z},$$

ubi binis litteris β et γ omnes possibiles valores tribui sunt censendae, quas autem nullo modo sub quapiam functione definita complecti licet. Quo hoc clarius appareat ponamus $z = lp$, $y = lq$ et $x = lr$, ut fiat

$$V = p^{\sqrt{\beta\gamma}} q^{\beta} r^{\gamma}.$$

Quod si iam hic litteris α, β, γ tantum valores integros tribuantur, prodibit talis aequatio:

$$V = \mathfrak{A}pqr + \mathfrak{B}p^{\sqrt{2}}q^2r + \mathfrak{C}p^{\sqrt{3}}q^3r + \text{etc.},$$

quos diversos terminos sub nulla certa functione complecti licet.

14. Interim tamen, si aequatio vicaria resolutionem in duos factores, etsi non simplices, admittat, tum integrale aequationis propositae ad integrationem duarum aequationum inferioris gradus reducitur. Quodsi enim aequatio differentialis proposita ascendat ad gradum differentialium $m = \mu + \nu$, eiusque forma vicaria habeat duos factores, alterum gradus μ, alterum vero gradus ν, si ex his vicissim formentur aequationes differentiales, altera ad gradum μ, altera ad gradum ν assurget. Ponamus ergo integrationem prioris praebere $V = P$, posterioris vero $V = Q$, atque manifestum est, ipsius aequationis propositae integrale fore $AP + BQ$; atque adeo, si illa integralia fuerint completa, etiam hoc erit completum[1]).

15. Quae hactenus sunt tradita, proprie quidem ad functiones trium variabilium z, y, x sunt accomodata: facile autem intelligitur, eadem praecepta pariter locum obtinere tam pro paucioribus, quam pro pluribus variabilibus.

1) Quod in peculiaribus casibus falsum esse potest. H. D.

ANALYSIS FACILIS AEQUATIONEM RICCATIANAM PER FRACTIONEM CONTINUAM RESOLVENDI

Conventui exhibuit die 20. Martii 1780
Commentatio 751 indicis ENESTROEMIANI
Mémoires de l'académie des sciences de St-Pétersbourg 6 (1813/4), 1818, p. 12—29

1. Iam pridem equidem aequationis RICCATIANAE resolutionem per fractionem continuam tradidi[1]), sed usus sum methodo haud parum operosa, quae transformationes satis abstrusas requirebat. Nunc autem se mihi obtulit alia via longe facilior idem praestandi, quae cum ad fractionem continuam multo simpliciorem perducat, haud indigna mihi visa est, ut eam cum publico communicarem, praecipue cum insignes Geometrae hoc argumentum summo studio perscrutari coeperint.

2. Considero autem aequationem RICCATIANAM, sub hac forma expressam:

$$ady + yydx = x^{n-2}dx,$$

quam autem hoc modo repraesento:

$$ady - \frac{bydx}{x} + yydx = x^{n-2}dx,$$

quae quidem revera priore non est generalior; posito enim $y = x^{\frac{b}{a}}v$ fit

$$dy = x^{\frac{b}{a}}dv + \frac{b}{a}x^{\frac{b-a}{a}}vdx,$$

1) Vide p. 183. Vide quoque notam p. 184. H. D.

hoc modo novus terminus introductus iterum tolletur, orieturque sequens aequatio:

$$ax^{\frac{b}{a}}dv + x^{\frac{2b}{a}}vvdx = x^{n-2}dx.$$

Verum forma assumpta ad institutum nostrum imprimis est accommodata; statuo autem praeterea $y = \frac{z}{x}$, ponoque brevitatis gratia $a + b = c$, ut prodeat sequens aequatio:

$$axdz - czdx + zzdx = x^n dx.$$

3. Fiat nunc ista substitutio:

$$z = c + \frac{x^n}{p},$$

ita ut sit

$$dz = \frac{npx^{n-1}dx - x^n dp}{pp},$$

factaque substitutione, et sublatis fractionibus, pervenietur ad hanc aequationem:

$$axdp - (c + na)pdx + ppdx = x^n dx,$$

quae prorsus similis est praecedenti; tantum enim eo discrepat, quod in secundo termino loco c habeamus $c + na$.

4. In hac aequatione iam statuamus porro

$$p = c + na + \frac{x^n}{q},$$

et calculo evoluto perveniemus ad sequentem aequationem:

$$axdq - (c + 2na)qdx + qqdx = x^n dx,$$

quam quidem immediate ex praecedente deducere potuissemus, scilicet loco p scribendo q et coefficientem secundi termini, qui erat $c + na$, denuo quantitate na augendo.

5. Simili modo intelligitur, si hic porro statuamus

$$q = c + 2na + \frac{x^n}{r},$$

prodituram esse hanc aequationem:

$$axdr - (c + 3na)rdx + rrdx = x^n dx,$$

atque ulterius, si hic statuam

$$r = c + 3na + \frac{x^n}{s},$$

prodibit ista aequatio:

$$axds - (c + 4na)sdx + ssdx = x^n dx.$$

Sicque porro in infinitum progredi licet.

6. Quodsi iam loco litterarum p, q, r, s etc. valores successive substituamus, pro variabili z reperiemus sequentem fractionem continuam satis concinnam:

$$z = c + \cfrac{x^n}{c + na + \cfrac{x^n}{c + 2na + \cfrac{x^n}{c + 3na + \cfrac{x^n}{c + 4na + \cfrac{x^n}{c + 5na + \text{etc.}}}}}}$$

unde aequationis praecedentis, quae, ob $b = c - a$, erat

$$ady + (a - c)\frac{ydx}{x} + yydx = x^{n-2}dx,$$

valor y hac fractione continua exprimetur:

$$y = \frac{c}{x} + \cfrac{x^{n-1}}{c + na + \cfrac{x^n}{c + 2na + \cfrac{x^n}{c + 3na + \text{etc.}}}}$$

7. Hinc igitur pro ipsa aequatione primum proposita:

$$ady + yydx = x^{n-2}dx,$$

ubi scilicet est $c = a$, erit

$$y = \frac{a}{x} + \cfrac{x^{n-1}}{a(1 + n) + \cfrac{x^n}{a(1 + 2n) + \cfrac{x^n}{a(1 + 3n) + \text{etc.}}}}$$

8. Quodsi ergo vicissim proponatur ista fractio continua:

$$z = c + \cfrac{x^n}{c + na + \cfrac{x^n}{c + 2na + \cfrac{x^n}{c + 3na + \text{etc.}}}}$$

nunc certi sumus valorem ipsius z determinari per hanc aequationem differentialem:

$$axdz - czdx + zzdx = x^n dx,$$

cuius ergo integrale, rite sumptum et ad hunc casum accommodatum, praebebit valorem illius fractionis continuae; scilicet integrale ita definiri debet, ut, posito $x = 0$, fiat $z = c$, si quidem n fuerit numerus positivus; at si fuerit negativus, prodire debet $z = c$, posito $x = \infty$.

9. Hinc plurima egregia consectaria deduci possunt pro casibus, quibus aequatio differentialis proposita integrationem admittit. Ita pro aequatione primum assumpta, ubi est $c = a$, si ponamus $n = 2$, erit $ady + yydx = dx$, unde colligitur $dx = \frac{ady}{1 - yy}$, cuius integrale est

$$x + \alpha = \tfrac{1}{2} a l \frac{1 + y}{1 - y},$$

et ad numeros ascendendo $\varDelta e^{\frac{2x}{a}} = \frac{1 + y}{1 - y}$, unde vicissim erit

$$y = \frac{\varDelta e^{\frac{2x}{a}} - 1}{\varDelta e^{\frac{2x}{a}} + 1}, \quad \text{hincque} \quad z = xy = x \frac{(\varDelta e^{\frac{2x}{a}} - 1)}{\varDelta e^{\frac{2x}{a}} + 1},$$

qui valor complectetur summam huius fractionis continuae:

$$z = a + \cfrac{xx}{3a + \cfrac{xx}{5a + \cfrac{xx}{7a + \cfrac{xx}{9a + \text{etc.}}}}}$$

quae cum praebeat $z = a$, sumpto $x = 0$, inde valor constantis $\varDelta$ rite determinari potest.

10. Cum igitur, ob $z = xy$, invenerimus

$$z = \frac{(\varDelta e^{\frac{2x}{a}} - 1)x}{\varDelta e^{\frac{2x}{a}} + 1},$$

constans $\varDelta$ ita determinari debet, ut, posito $x = 0$, fiat $z = a$. Hoc autem casu tota haec formula evanescit, nisi etiam eius denominator simul evanescat, quod fieri nequit, nisi sumatur $\varDelta = -1$. Cum igitur sit

$$z = -x\frac{(1 + e^{\frac{2x}{a}})}{1 - e^{\frac{2x}{a}}} = x\frac{(1 + e^{\frac{2x}{a}})}{e^{\frac{2x}{a}} - 1},$$

sumpto x quasi infinite parvo fiet $e^{\frac{2x}{a}} = 1 + \frac{2x}{a}$, quamobrem hoc casu habebimus

$$z = x\frac{\left(2 + \frac{2x}{a}\right)}{\frac{2x}{a}} = a + x,$$

ideoque facto $x = 0$, fiet $z = a$, prorsus uti requiritur. Quocirca nostrae fractionis continuae hic propositae valor erit

$$z = x\frac{(1 + e^{\frac{2x}{a}})}{e^{\frac{2x}{a}} - 1},$$

qui etiam hoc modo repraesentari potest

$$z = x\frac{(e^{\frac{x}{a}} + e^{\frac{-x}{a}})}{e^{\frac{x}{a}} - e^{\frac{-x}{a}}},$$

cuius fractionis numerator, facta evolutione, fit

$$2\left(1 + \frac{1}{2}\frac{xx}{aa} + \frac{1}{24}\frac{x^4}{a^4} + \frac{1}{720}\frac{x^6}{a^6} + \text{etc.}\right).$$

Denominator vero erit

$$\frac{2x}{a}\left(1 + \frac{1}{6}\frac{xx}{aa} + \frac{1}{120}\frac{x^4}{a^4} + \frac{1}{5040}\frac{x^6}{a^6} + \text{etc.}\right),$$

unde nanciscimur

$$z = \frac{1 + \frac{xx}{2aa} + \frac{x^4}{24a^4} + \frac{x^6}{720a^6} + \text{etc.}}{\frac{1}{a}\left(1 + \frac{1}{6}\frac{xx}{aa} + \frac{1}{120}\frac{x^4}{a^4} + \frac{1}{5040}\frac{x^6}{a^6} + \text{etc.}\right)}.$$ [1)]

11. Quodsi ergo sumamus $a = 1$, ut habeamus hanc fractionem continuam:

$$z = \frac{x(e^x + e^{-x})}{e^x - e^{-x}} = 1 + \cfrac{xx}{3 + \cfrac{xx}{5 + \cfrac{xx}{7 + \text{etc.}}}}$$

eadem fractio continua quoque ita exprimitur:

$$\frac{1 + \frac{1}{2}xx + \frac{1}{24}x^4 + \frac{1}{720}x^6 + \text{etc.}}{1 + \frac{1}{6}xx + \frac{1}{120}x^4 + \frac{1}{5040}x^6 + \text{etc.}},$$

quae ambae series eo magis convergunt, quo minor valor ipsi x tribuatur. Scilicet si ponatur $x = 1$, huius fractionis continuae:

$$1 + \cfrac{1}{3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \text{etc.}}}}}$$

valor erit

$$\frac{e^1 + e^{-1}}{e^1 - e^{-1}} = \frac{ee + 1}{ee - 1}.$$

12. Verum etiam ipsa haec fractio continua vehementer convergit. Si enim indices 1, 3, 5, 7 etc. ordine disponamus, et more solito fractiones subscribamus, eae continuo propius ad verum valorem huius formae procedunt; haec autem erit huius operationis species:

$$\begin{array}{ccccccc} 1, & 3, & 5, & 7, & 9, & 11 & \text{etc.} \\ \frac{1}{0}, & \frac{1}{1}, & \frac{4}{3}, & \frac{21}{16}, & \frac{151}{115}, & \frac{1380}{1051} & \text{etc.} \end{array}$$

1) Vide notam p. 174. Cf. quoque Commentationem 750: *Commentatio in fractionem continuam, qua Illustris* LA GRANGE *potestates binomiales expressit.* Mém. Pétersb. 6, 1818, p. 11. *LEONHARDI EULERI Opera omnia*, series I, vol. 16, p. 232. H. D.

Quo autem pateat, quam vehementer hae aequalitates ad veritatem convergant, considerentur differentiae inter terminos contiguos, quae erunt alternatim positivae et negativae, atque ita ordine procedunt:

$$\infty, \quad -\frac{1}{3}, \quad +\frac{1}{3\cdot 16}, \quad \frac{-1}{16\cdot 115}, \quad +\frac{1}{115\cdot 1051}, \quad \frac{-1}{1051\cdot 11676}.$$

Nunc igitur certo scimus errorem postremae nostrae fractionis $\frac{1380}{1051}$ certe minorem esse, quam $\frac{1}{1051\cdot 11676}=\frac{1}{12271476}$. Minus ergo discrepat a valore vero $\frac{ee+1}{ee-1}$.

13. Hic autem quaestio magni momenti se offert, quantus futurus sit valor fractionis continuae pro z inventae:

$$z=a+\cfrac{xx}{3a+\cfrac{xx}{5a+\text{etc.}}}$$

si loco xx scribamus $-tt$, ita ut sit $x=t\sqrt{-1}$. Tum autem habebimus

$$z=t\sqrt{-1}\,\frac{\left(e^{\frac{t\sqrt{-1}}{a}}+e^{\frac{-t\sqrt{-1}}{a}}\right)}{e^{\frac{t\sqrt{-1}}{a}}-e^{\frac{-t\sqrt{-1}}{a}}},$$

unde ergo imaginaria extrudere oportet. Cum autem sit:

$$e^{\varphi\sqrt{-1}}=\cos.\varphi+\sqrt{-1}\sin.\varphi$$

et

$$e^{-\varphi\sqrt{-1}}=\cos.\varphi-\sqrt{-1}\sin.\varphi,$$

erit valor quaesitus

$$z=\frac{t}{\operatorname{tg}.\frac{t}{a}}=t\cot.\frac{t}{a},$$

qui ergo est valor huius fractionis continuae:

$$z=a-\cfrac{tt}{3a-\cfrac{tt}{5a-\cfrac{tt}{7a-\text{etc.}}}}$$

Ex illa autem aequatione, posito $t=0$, manifesto fit $z=a$.

14. Hoc casu, quo harum fractionum continuarum valores actu determinare licuit sive per formulas exponentiales, sive per arcus circulares, prima aequatio proposita integrationem admisit. Quia autem infiniti alii casus integrabiles dantur, examinemus adhuc casum $n = -2$ et prima aequatio erit

$$ady + yydx = \frac{dx}{x^4},$$

ubi scilicet est $c = a$, pro qua integranda statuamus

$$y = \frac{\alpha}{x} + \frac{\beta}{xx},$$

qua substitutione facta prodibit haec aequatio:

$$\left.\begin{aligned} &-\frac{\alpha a}{xx} - \frac{2\beta a}{x^3} + \frac{\beta\beta}{x^4} \\ &+\frac{\alpha\alpha}{xx} + \frac{2\alpha\beta}{x^3} \end{aligned}\right\} = \frac{1}{x^4},$$

ubi igitur requiritur ut sit $\alpha = a$ et $\beta\beta = 1$, ideoque vel $\beta = +1$, vel $\beta = -1$. Sicque geminum habemus valorem pro y, quorum

$$\text{alter est } y = \frac{a}{x} + \frac{1}{xx}, \quad \text{alter vero } y = \frac{a}{x} - \frac{1}{xx}.$$

15. Haec autem integralia tantum sunt particularia; pro nostro scopo autem integrale completum nosse oportet, ut scilicet constans arbitraria ad statum quaestionis accommodari possit. Quia hic vero binos habemus valores particulares, eos ponamus

$$\frac{a}{x} + \frac{1}{xx} = p \quad \text{et} \quad \frac{a}{x} - \frac{1}{xx} = q\text{[1]},$$

ita ut revera sit

$$adp + ppdx = \frac{dx}{x^4} \quad \text{et} \quad adq + qqdx = \frac{dx}{x^4},$$

harum utraque subtrahatur ab ipsa aequatione integranda, et orientur hae duae aequalitates:

$$a(dy - dp) + dx(yy - pp) = 0$$

et

$$a(dy - dq) + dx(yy - qq) = 0,$$

1) Cf. L. Euleri Commentationem 734, § 6. Vide p. 382. H. D.

quarum illa per $y-p$, haec vero per $y-q$ divisa, praebet has aequationes:

$$a\frac{(dy-dp)}{y-p}+dx(y+p)=0$$

et

$$a\frac{(dy-dq)}{y-q}+dx(y+q)=0,$$

quarum haec, ab illa subtracta, relinquit

$$a\frac{(dy-dp)}{y-p}-a\frac{(dy-dq)}{y-q}+dx(p-q)=0,$$

cuius ergo integrale est

$$al(y-p)-al(y-q)+\int dx(p-q)=\text{const.}$$

Quia igitur est $p-q=\frac{2}{xx}$, erit

$$\int dx(p-q)=-\frac{2}{x},$$

sicque habebimus

$$al\frac{y-p}{y-q}=C+\frac{2}{x},$$

unde porro colligitur

$$\frac{y-p}{y-q}=\Delta e^{\frac{2}{ax}}.$$

16. Ponamus hic brevitatis gratia $e^{\frac{2}{ax}}=\omega$, unde colligitur

$$y=\frac{p-\Delta\omega q}{1-\Delta\omega},$$

et pro p et q, restitutis valoribus, habebitur

$$y=\frac{1+ax+(1-ax)\Delta\omega}{xx(1-\Delta\omega)}.$$

Hincque porro erit

$$z=\frac{1+ax+(1-ax)\Delta\omega}{x(1-\Delta\omega)}=\frac{1+\Delta\omega}{x(1-\Delta\omega)}+a.$$

Quia vero est $c = a$ et $n = -2$, fractio nostra continua pro z inventa erit

$$z = a + \cfrac{x^{-2}}{-a + \cfrac{x^{-2}}{-3a + \cfrac{x^{-2}}{-5a + \cfrac{x^{-2}}{-7a + \text{etc.}}}}}$$

17. Quia nunc in hac expressione, sumpto $x = \infty$, fit $z = a$, constans illa arbitraria Δ convenienter debet determinari, quamobrem, posito $x = \infty$, fieri debet

$$\frac{1 + \Delta\omega}{x(1 - \Delta\omega)} = 0,$$

hic autem est $\omega = e^{\frac{2}{ax}}$, qui valor, sumpto $x = \infty$, evadit $= 1$, quo observato fieri debet $0 = \frac{1 + \Delta}{x(1 - \Delta)}$. Quia vero hinc Δ nondum determinatur, haec investigatio accuratius institui debet; sumpto scilicet x praemagno, ex fractione continua, per unum membrum continuata, fit

$$z = a - \frac{1}{axx},$$

cui ergo aequari debet haec expressio

$$a + \frac{1 + \Delta\omega}{x(1 - \Delta\omega)},$$

ideoque esse debet

$$\frac{1 + \Delta\omega}{1 - \Delta\omega} = \frac{-1}{ax},$$

unde colligitur

$$\Delta\omega = \frac{1 + ax}{1 - ax} = \Delta\left(1 + \frac{2}{ax}\right),$$

sicque erit

$$\Delta = \frac{ax(1 + ax)}{(1 - ax)(2 + ax)}.$$

Posito hic $x = \infty$ prodit $\Delta = -1$, ideoque erit

$$z = a + \frac{1 - \omega}{x(1 + \omega)},$$

existente $\omega = e^{\frac{2}{ax}}$, cuius expressionis valor etiam per series communes satis commode exprimi poterit. Cum enim sit $\omega = e^{\frac{2}{ax}}$, fractio $\frac{1-\omega}{1+\omega}$ hac forma exprimi poterit:

$$\frac{e^{\frac{-1}{ax}} - e^{\frac{1}{ax}}}{e^{\frac{-1}{ax}} + e^{\frac{1}{ax}}},$$

cuius numerator in hanc seriem evolvitur:

$$-2\left(\frac{1}{ax} + \frac{1}{6a^3x^3} + \frac{1}{120a^5x^5} + \text{etc.}\right),$$

denominator vero erit

$$2\left(1 + \frac{1}{2a^2x^2} + \frac{1}{24a^4x^4} + \frac{1}{720a^6x^6} + \text{etc.}\right).$$

Tum manifestum est, fore

$$z = a - \frac{\left(\frac{1}{ax} + \frac{1}{6a^3x^3} + \frac{1}{120a^5x^5} + \text{etc.}\right)}{x\left(1 + \frac{1}{2a^2x^2} + \frac{1}{24a^4x^4} + \frac{1}{720a^6x^6} + \text{etc.}\right)}.$$

18. Praeterea cum in fractionibus primam sequentibus partes absolutae sint negativae, eas facile in positivas transmutare licet. Ponatur enim $z = a + \frac{x^{-2}}{v}$ ut sit

$$v = -a + \cfrac{x^{-2}}{-3a + \cfrac{x^{-2}}{-5a + \cfrac{x^{-2}}{-7a + \text{etc.}}}}$$

unde fit

$$-v = a + \cfrac{x^{-2}}{3a + \cfrac{x^{-2}}{5a + \cfrac{x^{-2}}{7a + \text{etc.}}}}$$

Quare cum sit $z = a - \frac{x^{-2}}{-v}$, loco $-v$ substituto valore habebimus

$$z = a - \cfrac{x^{-2}}{a + \cfrac{x^{-2}}{3a + \cfrac{x^{-2}}{5a + \cfrac{x^{-2}}{7a + \text{etc.}}}}}$$

19. Consideremus nunc iterum fractionem continuam generalem, pro z inventam, quae erat

$$z = c + \cfrac{x^n}{c + na + \cfrac{x^n}{c + 2na + \cfrac{x^n}{c + 3na + \cfrac{x^n}{c + 4na + \text{etc.}}}}}$$

ubi valores ipsius z ex hac aequatione differentiali:

$$ax\,dz - cz\,dx + zz\,dx = x^n\,dx$$

determinantur. Iam in fractione continua omnes partes absolutae, quae sunt c, $c + na$, $c + 2na$, $c + 3na$ etc., progressionem arithmeticam constituunt secundum differentiam na crescentem, at vero numeratores omnes sunt inter se aequales, scilicet x^n. Hinc ergo vicissim, quoties talis fractio continua occurrit, eius valor per aequationem differentialem ad genus RICCATIANUM pertinentem determinari poterit, id quod in sequente problemate accuratius prosequamur.

PROBLEMA

20. Proposita in genere hac fractione continua[1]):

$$z = m + \cfrac{\varDelta}{m + n + \cfrac{\varDelta}{m + 2n + \cfrac{\varDelta}{m + 3n + \cfrac{\varDelta}{m + 4n + \text{etc.}}}}}$$

cuius partes absolutae in progressione arithmetica progrediantur, numeratores vero omnes sint inter se aequales; eius valorem z ad resolutionem aequationis RICCATIANAE reducere.

SOLUTIO

Si haec forma cum modo ante allata comparetur, evidens est, eam in hanc converti, si statuamus $a = 1$ et $c = m$, tum vero potestati x^n tribui debet valor $\varDelta$, quod quidem nonnisi integratione peracta fieri licet. Quamobrem valor quaesitus pro z ex hac aequatione differentiali:

1) Cf. p. 184. H. D.

$$xdz - mzdx + zzdx = x^n dx,$$

derivari debet.

21. Ut nunc hanc aequationem ad consuetam formam RICCATIANAE reducamus, statuamus $z = x^m v$, ut scilicet hoc modo aequatio ad tres terminos reducatur:

$$x^{m+1} dv + vvx^{2m} dx = x^n dx,$$

quae, per x^{m+1} divisa, abit in hanc:

$$dv + vvx^{m-1} dx = x^{n-m-1} dx,$$

quam formam ut penitus ad RICCATIANAM reducamus, ponamus $x^m = t$, ut fiat $x^{m-1}dx = \frac{dt}{m}$ et $x = t^{\frac{1}{m}}$, ideoque

$$dx = \frac{1}{m} t^{\frac{1-m}{m}} dt \quad \text{et} \quad x^{n-m-1} = t^{\frac{n-m-1}{m}},$$

quibus substitutis aequatio nostra fiet

$$dv + \frac{vvdt}{m} = \frac{1}{m} t^{\frac{n-2m}{m}} dt, \quad \text{sive} \quad mdv + vvdt = t^{\frac{n-2m}{m}} dt,$$

quae est ipsa aequatio RICCATO debita.

22. Perpendamus nunc ante omnia casus, quibus haec aequatio resolutionem admittit, qui sunt quando in termino ad dextram posito exponens $\frac{n-2m}{m}$ in hac forma continetur $\frac{-4i}{2i+1}$, denotante i numerum quemcunque integrum, sive positivum, sive negativum. Ponamus igitur $\frac{n-2m}{m} = -\frac{4i}{2i+1}$, et utrinque binarium addendo habebimus $\frac{n}{m} = \frac{2}{2i+1}$, unde sumi poterit $m = 2i+1$ et $n = 2$. Hinc patet sumpto $n = 2$, quoties fuerit m numerus impar quicunque, sive positivus, sive negativus, valorem fractionis continuae actu assignari posse, quod ergo contingit si fuerit

$$z = 2i + 1 + \cfrac{\varDelta}{2i + 3 + \cfrac{\varDelta}{2i + 5 + \cfrac{\varDelta}{2i + 7 + \text{etc.}}}}$$

Ubi quidem notandum est absoluta integratione fieri debere $x^n = \varDelta$, hoc est $xx = \varDelta$, existente $x^m = t = x^{2i+1}$.

23. Quoties autem fractio continua proposita non his conditionibus continetur, tum etiam per methodos adhuc cognitas finito modo neutiquam exprimi poterit, sed contenti esse debemus eius valorem ad aequationem RICCATIANAM perduxisse, quippe cuius resolutio per series infinitas satis commode exhiberi potest, id quod unico exemplo declarabimus.

EXEMPLUM

24. Proposita nobis sit haec fractio continua:

$$z = 1 + \cfrac{\varDelta}{2 + \cfrac{\varDelta}{3 + \cfrac{\varDelta}{4 + \text{etc.}}}}$$

Hic ergo erit $m = 1$ et $n = 1$, atque valor ipsius z ex hac aequatione differentiali quaeri oportet:

$$xdz - zdx + zzdx = xdx,$$

hacque aequatione resoluta loco x scribi oportebit $\varDelta$, unde patet integrationem ita institui debere, ut posito $x = 0$ fiat $z = 1$.

Ponatur nunc $z = xv$, ut oriatur haec aequatio:

$$dv + vvdx = \frac{dx}{x},$$

quae ut commode in seriem resolvatur, ponamus $v = \frac{du}{udx}$, et sumpto elemento dx constante fit

$$xddu - udx^2 = 0,$$

quae iam facile in seriem resolvi poterit per potestates naturales ipsius x ascendentem. Statuatur ergo

$$u = ax^\lambda + bx^{\lambda+1} + cx^{\lambda+2} + dx^{\lambda+3} + \text{etc.}$$

eritque

$$\frac{ddu}{dx^2} = a\lambda(\lambda-1)x^{\lambda-2} + b(\lambda+1)\lambda x^{\lambda-1} + c(\lambda+2)(\lambda+1)x^\lambda + d(\lambda+3)(\lambda+2)x^{\lambda+1} + \text{etc.},$$

quae series aequari debet ipsi $\frac{u}{x}$, hoc est

$$ax^{\lambda-1} + bx^{\lambda} + cx^{\lambda+1} + dx^{\lambda+2} + ex^{\lambda+3} + \text{etc.}$$

Unde patet prioris seriei terminum $a\lambda(\lambda - 1)$ ad nihilum redigi debere, id quod duplici modo fieri debet, sumendo vel $\lambda = 0$ vel $\lambda = 1$.

25. Sumatur ergo primo $\lambda = 0$ et series nostrae coaequandae erunt:

$$\text{I. } 2c + 6dx + 12ex^2 + 20fx^3 + \text{etc.}$$

et

$$\text{II. } ax^{-1} + b + cx + dxx + ex^3 + \text{etc.};$$

erit igitur:

$$a = 0, \quad c = \frac{b}{1\cdot 2}, \quad d = \frac{c}{2\cdot 3} = \frac{b}{1\cdot 2\cdot 2\cdot 3},$$
$$e = \frac{d}{3\cdot 4} = \frac{b}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4}, \quad f = \frac{e}{4\cdot 5} = \frac{b}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4\cdot 4\cdot 5},$$

quamobrem nostra series pro u, sumpto $b = 1$ erit

$$u = x + \frac{1}{1\cdot 2}xx + \frac{1}{1\cdot 2^2\cdot 3}x^3 + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4}x^4 + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5}x^5 + \text{etc.}$$

26. Simili modo evolvamus casum $\lambda = 1$, ac series pro $\frac{ddu}{dx^2}$ inventa erit

$$1\cdot 2b + 2\cdot 3cx + 3\cdot 4dx^2 + 4\cdot 5ex^3 + 5\cdot 6fx^4 + \text{etc.},$$

cui aequalis esse debet

$$\frac{u}{x} = a + bx + cxx + dx^3 + ex^4 + fx^5 + \text{etc.},$$

unde fit:

$$a = 2b, \quad b = \frac{1}{1\cdot 2}a, \quad c = \frac{b}{2\cdot 3} = \frac{a}{1\cdot 2^2\cdot 3},$$
$$d = \frac{c}{3\cdot 4} = \frac{a}{1\cdot 2^2\cdot 3^2\cdot 4}, \quad e = \frac{d}{4\cdot 5} = \frac{a}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5} \text{ etc.}$$

Hinc ergo sumpto $a = 1$, pro u habebimus hanc seriem:

$$u = x + \frac{1}{1\cdot 2}xx + \frac{1}{1\cdot 2^2\cdot 3}x^3 + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4}x^4 + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5}x^5 + \text{etc.},$$

quae cum praecedente prorsus congruit.

27. Invento iam valore litterae u, ob $v = \frac{du}{udx}$ erit nunc

$$v = \frac{1 + x + \frac{x^2}{1\cdot 2^2} + \frac{x^3}{1\cdot 2^2\cdot 3^2} + \frac{x^4}{1\cdot 2^2\cdot 3^2\cdot 4^2} + \text{etc.}}{x + \frac{xx}{1\cdot 2} + \frac{x^3}{1\cdot 2^2\cdot 3} + \frac{x^4}{1\cdot 2^2\cdot 3^2\cdot 4} + \frac{x^5}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5} + \text{etc.}}.$$

Quocirca ipse valor fractionis continuae propositae erit

$$z = \frac{1 + x + \frac{xx}{1\cdot 2^2} + \frac{x^3}{1\cdot 2^2\cdot 3^2} + \frac{x^4}{1\cdot 2^2\cdot 3^2\cdot 4^2} + \text{etc.}}{1 + \frac{x}{1\cdot 2} + \frac{xx}{1\cdot 2^2\cdot 3} + \frac{x^3}{1\cdot 2^2\cdot 3^2\cdot 4} + \frac{x^4}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5} + \text{etc.}},$$

ubi commode evenit, ut sumpto $x = 0$ fiat $z = 1$. Quamobrem, si faciamus $x = \Delta$, verus valor fractionis continuae propositae erit

$$z = \frac{1 + \Delta + \frac{\Delta^2}{1\cdot 2^2} + \frac{\Delta^3}{1\cdot 2^2\cdot 3^2} + \frac{\Delta^4}{1\cdot 2^2\cdot 3^2\cdot 4^2} + \text{etc.}}{1 + \frac{\Delta}{1\cdot 2} + \frac{\Delta^2}{1\cdot 2^2\cdot 3} + \frac{\Delta^3}{1\cdot 2^2\cdot 3^2\cdot 4} + \frac{\Delta^4}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5} + \text{etc.}}.$$

COROLLARIUM

28. Sumamus $\Delta = 1$, ita ut proponatur haec fractio continua:

$$z = 1 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{4 + \cfrac{1}{5 + \text{etc.}}}}}$$

atque valor ipsius z etiam sequenti modo exprimetur:

$$z = \frac{2 + \frac{1}{1\cdot 2^2} + \frac{1}{1\cdot 2^2\cdot 3^2} + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4^2} + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5^2} + \text{etc.}}{1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2^2\cdot 3} + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4} + \frac{1}{1\cdot 2^2\cdot 3^2\cdot 4^2\cdot 5} + \text{etc.}},$$

cuius valor in fractionibus decimalibus satis commode exprimi poterit. Reperitur autem numerator $= 2{,}279584$[1]) et denominator $= 1{,}590635$, unde porro colligitur valor ipsius $z = 1{,}432490$.

1) Editio princeps: 8 loco 9. Correxit H. D.

29. Potest vero etiam iste valor ex ipsa fractione continua deduci. Cum enim omnes numeratores sint $= 1$, partes absolutae tanquam indices ordine scribantur, et more solito fractiones subscribantur, ut sequitur:

$$\begin{array}{cccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7 & \text{etc.} \\ \dfrac{1}{0}, & \dfrac{1}{1}, & \dfrac{3}{2}, & \dfrac{10}{7}, & \dfrac{43}{30}, & \dfrac{225}{157}, & \dfrac{1393}{972} & \text{etc.} \end{array}$$

quae fractiones eo propius ad veritatem accedent, quo ulterius continuentur.

SOLUTIO PROBLEMATIS AD ANALYSIN INFINITORUM INDETERMINATORUM REFERENDI

Conventui exhibita die 20. Augusti 1781
Commentatio 779 indicis ENESTROEMIANI
Mémoires de l'académie des sciences de St.-Pétersbourg 11, 1830, p. 92—94

Problema, cuius hic solutionem tradere animus est, ita enunciatur: Propositis quotcunque functionibus p, q, r, s, t etc. eiusdem variabilis v, invenire functionem x, ita comparatam, ut omnes, quos ecce formulae differentiales:

$$p\,\partial x,\ q\,\partial x,\ r\,\partial x,\ s\,\partial x,\ t\,\partial x \text{ etc.}$$

evadant integrabiles[1]), cuius solutio, breviter exposita ita se habet.

Postquam functiones datae pro lubitu certo ordine fuerint dispositae, veluti hoc modo: p, q, r, s etc., ex iis deriventur sequentes functiones primi gradus:

$$q' = \frac{\partial q}{\partial p},\quad r' = \frac{\partial r}{\partial p},\quad s' = \frac{\partial s}{\partial p},\quad t' = \frac{\partial t}{\partial p} \text{ etc.}$$

Ex his simili modo formentur sequentes secundi gradus:

$$r'' = \frac{\partial r'}{\partial q'},\quad s'' = \frac{\partial s'}{\partial q'},\quad t'' = \frac{\partial t'}{\partial q'} \text{ etc.,}$$

quarum numerus iam unitate minor est quam numerus praecedentium. Hinc porro deducantur functiones gradus tertii, quae erunt $s''' = \dfrac{\partial s''}{\partial r''}$, $t''' = \dfrac{\partial t''}{\partial r''}$ etc.,

1) Cf. Commentationes 622 et 650 p. 197 et p. 210. Vide quoque notam p. 198. H. D.

quarum numerus iterum unitate minor est praecedentium numero, et ita porro, ita ut si functionum propositarum numerus fuerit 5, ultimo sit $t^{IV} = \frac{\partial t'''}{\partial s'''}$.

Iam simili ratione ex functione quaesita x formemus alias per similes gradus, quae sint:

$$x = \frac{\partial x'}{\partial p}, \quad x' = \frac{\partial x''}{\partial q'}, \quad x'' = \frac{\partial x'''}{\partial r''}, \quad x''' = \frac{dx^{IV}}{\partial s'''} \text{ etc.},$$

unde vicissim habebimus sequentes determinationes:

$$\partial x' = x\,\partial p, \ \partial x'' = x'\,\partial q', \ \partial x''' = x''\,\partial r'' \text{ etc.}$$

His formulis cum praecedentibus coniunctis [habebimus] sequentes determinationes seu reductiones formularum primi gradus:

$$q'\,\partial x' = x\,\partial q, \ r'\,\partial x' = x\,\partial r, \ s'\,\partial x' = x\,\partial s, \ t'\,\partial x' = x\,\partial t \text{ etc.}$$

Eodem modo formulae secundi gradus ad primum reducentur, cum sit etiam

$$r''\,\partial x'' = x'\,\partial r', \ s''\,\partial x'' = x'\,\partial s', \ t''\,\partial x'' = x'\,\partial t' \text{ etc.};$$

porro formulae tertii gradus ad secundum, ob

$$s'''\,\partial x''' = x''\,\partial s'', \ t'''\,\partial x''' = x''\,\partial t'' \text{ etc.}$$

et ita porro.

Cum iam in ordine litterarum x, x', x'', x''', x^{IV} etc. perventum fuerit ad ultimam, quae casu quinque functionum erit x^V, pro ea accipiatur ad lubitum functio quaecunque ipsius v, quae sit V, ita ut habeamus $x^V = V$, atque hinc praecedentes omnes sponte determinabuntur, cum sit:

$$x^{IV} = \frac{\partial x^V}{\partial t^{IV}}, \quad x''' = \frac{\partial x^{IV}}{\partial s'''}, \quad x'' = \frac{\partial x'''}{\partial r''}, \quad x' = \frac{\partial x''}{\partial q'}, \quad x = \frac{\partial x'}{\partial p}.$$

His iam valoribus inventis integralia omnium formularum, quae requiruntur, ita se habebunt:

$$\begin{aligned}
\int p\,\partial x &= px - x' \\
\int q\,\partial x &= qx - q'x' + x'' \\
\int r\,\partial x &= rx - r'x' + r''x'' - x''' \\
\int s\,\partial x &= sx - s'x' + s''x'' - s'''x''' + x^{IV} \\
\int t\,\partial x &= tx - t'x' + t''x'' - t'''x''' + t^{IV}x^{IV} - x^{V} \\
&\text{etc.}
\end{aligned}$$

quarum formularum veritas per differentiationem sponte elucet.

Ex his iam tota Problematis solutio est manifesta, sive numerus functionum propositarum fuerit maior sive minor. Quovis enim casu valor ipsius x semper per differentialia eiusdem gradus, quot fuerint functiones propositae, exprimatur, ita ut, si duae tantum proponantur, quantitas x ad differentialia secundi gradus ascendet, si ternae, ad differentialia tertii ordinis et ita porro.

Hic denique observandum occurrit, prouti functiones propositae alio atque alio modo disponantur, ad integralia maxime diversa perventum iri, quae tamen omnia inter se convenire necesse est, siquidem quilibet ordo ad solutionem generalem perducit. Revera autem quaelibet horum integralium forma ad quamlibet aliam reduci potest, si loco functionis V assumamus TV. Semper enim littera T ita determinari potest, ut quaelibet forma integralium ad quamlibet aliam reducatur.

SOLUTIO PROBLEMATIS ANALYTICI DIFFICILLIMI

Conventui exhibita die 19. Augusti 1782

Commentatio 784 indicis ENESTROEMIANI

Mémoires de l'académie des sciences de St.-Pétersbourg **11**, 1830, p. 125—130

1. Si p, q et P, Q denotent functiones homogeneas nullius dimensionis binarum variabilium x et y datas et proposita fuerit haec formula differentialis $\partial v = \frac{p\,\partial x + \Pi q\,\partial y}{\Pi P + Q} x^{n-1}$, in quam ingreditur functio indeterminata Π, quam ita determinari oportet, ut integratio succedat. Huiusmodi formulae mihi se obtulerunt, cum nuper[1]) problema de traiectoriis orthogonalibus ad superficies translatum perscrutarer, atque evidens est, hanc quaestionem maxime esse arduam et summam sagacitatem in evolvendis functionibus duarum variabilium requirere, in quo negotio geometrae nunc quidem plurimum sunt occupati.

2. Quod si igitur statuamus $y = tx$, erunt litterae p, q, P, Q functiones datae ipsius $t = \frac{y}{x}$, et quia potestas indefinita ipsius x est adiuncta, haec formula omnes complectitur casus, quibus tam numerator quam denominator sunt functiones homogeneae ipsarum x et y. Positione igitur $y = tx$ tota formula ad has duas variabiles x et t reducitur. Quemadmodum igitur functio illa indefinita Π determinari debeat, hic nunc accuratius investigemus.

3. Ac primo quidem statuamus $\Pi = \frac{\Theta Q + \Delta}{1 - \Theta P}$, ubi scilicet binas novas functiones incognitas Δ et Θ introducimus, et facta hac substitutione formula nostra in sequentes duas partes discerpetur:

ationem 757 voluminis I 28 eodem anno 1782 conscriptam atque Com-
ultimi EULERI sunt. H. D.

$$\partial v = \frac{p\partial x + \Delta q\partial y}{\Delta P + Q} x^{n-1} + \frac{\Theta(Qq\partial y - Pp\partial x)}{\Delta P + Q} x^{n-1},$$

quarum priorem brevitatis gratia per ∂u, posteriorem vero per ∂w designabo, atque binas litteras Δ et Θ ita definire conabor, ut utraque pars integrationem admittat.

4. Nunc loco ∂y scribamus eius valorem $t\partial x + x\partial t$ atque pars prior induet hanc formam:

$$\partial u = \frac{x^{n-1}\partial x(p + \Delta qt)}{\Delta P + Q} + \frac{x^n \Delta q\partial t}{\Delta P + Q},$$

quae quo facilius tractari possit, statuatur

$$\frac{p + \Delta qt}{\Delta P + Q} = \Sigma,$$

unde fit

$$\Delta = \frac{\Sigma Q - p}{qt - \Sigma P}$$

et pro altero membro fit

$$\frac{\Delta q}{\Delta P + Q} = \frac{\Sigma Qq - pq}{Qqt - Pp},$$

sicque habebitur

$$\partial u = \Sigma x^{n-1}\partial x + \frac{x^n q\partial t(\Sigma Q - p)}{Qqt - Pp}.$$

Quamobrem, si Σ tantum involvat variabilem t, integrale aliam formam habere nequit nisi hanc:

$$u = \frac{1}{n}\Sigma x^n;$$

tum autem esse debet

$$\partial \Sigma = \frac{nq\partial t(\Sigma Q - p)}{Qqt - Pp},$$

cuius aequationis resolutio, quia Σ non ultra unam dimensionem ascendit, est facilis.

5. Quo autem integrale commodius exprimatur, ponamus

$$\frac{Qq\partial t}{Qqt - Pp} = \frac{\partial s}{s},$$

hincque integrale erit

$$\frac{\Sigma}{s^n} = -n\int \frac{pq\partial t}{s^n(Qqt - Pp)}.$$

Ergo quia ex praecedente positione est

$$\frac{q\partial t}{Qqt - Pp} = \frac{\partial s}{Qs}, \quad \text{erit} \quad \Sigma = -ns^n \int \frac{p\partial s}{Qs^{n+1}},$$

quam integrationem ut concessam assumamus et statuamus

$$\int \frac{p\partial s}{Qs^{n+1}} = T,$$

ita ut sit

$$\Sigma = -ns^n T,$$

hocque modo adepti sumus integrale prioris partis

$$u = -x^n s^n T,$$

qui valor ergo etiam praebet valorem quaesitum v pro casu, quo $\Theta = 0$.

6. Eodem modo evolvamus alteram partem, unde loco ∂y scripto valore $t\partial x + x\partial t$ prodit:

$$\partial w = \frac{\Theta x^{n-1}\partial x(Qqt - Pp) + \Theta x^n Qq\partial t}{(Qqt - Pp):(qt - \Sigma P)},$$

postquam scilicet loco Δ valorem ante inventum substituimus, qui erat

$$\Delta = \frac{\Sigma Q - p}{qt - \Sigma P}.$$

Hic igitur loco $\Theta(qt - \Sigma P)$ scribamus Φ, ut habeamus:

$$\partial w = \Phi\left(x^{n-1}\partial x + \frac{x^n\partial s}{s}\right) = \Phi\frac{x^{n-1}}{s}(s\partial x + x\partial s).$$

Unde patet integrale w fore functionem quamcunque ipsius sx, quam ita repraesentemus: $w = \varphi : xs$; sive, ponendo $xs = z$, si Z functionem quamcunque ipsius z denotet, habebitur $w = Z$; inde autem si ponatur $\partial Z = Z'\partial z$, erit

$$\Phi = \frac{Z's}{x^{n-1}}.$$

7. Inventa igitur utraque parte u et w, erit integrale quaesitum nostrae formulae

$$v = -Tz^n + Z,$$

qui valor praeter omnem expectationem tam simplex est inventus atque adeo facile ex ipsa formula proposita formari poterit, cum sit

$$\frac{\partial s}{s} = \frac{Qq\partial t}{Qqt - Pp} \quad \text{et} \quad T = \int \frac{p\partial s}{Qs^{n+1}},$$

hicque est valor generalissimus pro formula nostra proposita, siquidem loco Π successive valores hic assignati accipiantur, in quo negotio cum quaestio nostra potissimum versetur, operae pretium erit istum valorem evolvere.

8. Cum igitur primo posuerimus

$$\Pi = \frac{\Theta Q + \Delta}{1 - \Theta P},$$

deinde vero esset

$$\Delta = \frac{\Sigma Q - p}{qt - \Sigma P} \quad \text{et} \quad \Theta = \frac{\Phi}{qt - \Sigma p},$$

fiet

$$\Pi = \frac{-p + (\Phi + \Sigma)Q}{qt - (\Phi + \Sigma)P}.$$

Cum igitur sit

$$\Sigma = -ns^n T \quad \text{et} \quad \Phi = \frac{Z's}{x^{n-1}},$$

hincque pro quovis casu oblato valor debitus ipsi Π facile assignari potest.

ALIA SOLUTIO MULTO CONCINNIOR

9. Hic statim sine ulla praeparatione ipsa formula proposita, elidendo ∂y dat

$$\frac{x^{n-1}\partial x(p + \Pi qt) + x^n \Pi q \partial t}{\Pi P + Q} = \partial v.$$

Ponatur nunc

$$\frac{p + \Pi qt}{\Pi P + Q} = \Theta,$$

ut sit

$$\Pi = \frac{\Theta Q - p}{qt - \Theta P},$$

hincque porro

$$\frac{\Pi}{\Pi P + Q} = \frac{\Theta Q - p}{Qqt - Pp},$$

unde fit

$$\partial v = \Theta x^{n-1}\partial x + \frac{x^n q \partial t(\Theta Q - p)}{Qqt - Pp}.$$

10. Ponamus nunc ut supra

$$\frac{Qq\partial t}{Qqt - Pp} = \frac{\partial s}{s},$$

hocque modo fit

$$\partial v = \Theta x^{n-1}\partial x + \frac{x^n \partial s}{Qs}(\Theta Q - p).$$

Hinc autem facile conditio integrabilitatis obtineretur, verum nulla functio arbitraria praeterea in integrale introduceretur, quemadmodum solutio completa postulat. At vero singulari artificio etiam hinc integrale completum erui potest[1]), ponendo

$$\Theta = M + N.$$

Etiamsi enim haec positio nihil plane polliceri videatur, tamen ea totum negotium absolvetur. Hoc enim modo nostra formula distinguetur in duas partes, quarum utramque seorsim tractare licebit. Reperietur enim:

$$\partial v = Mx^{n-1}\partial x + x^n\frac{\partial s}{s}\left(M - \frac{p}{Q}\right) + N\frac{x^{n-1}}{s}(s\partial x + x\partial s).$$

11. Prioris partis litteram M involventis, siquidem M spectetur ut functio ipsius t tantum, integrale necessarium est $\frac{1}{n}Mx^n$; tum autem esse debet

$$\frac{\partial M}{n} = \frac{\partial s}{s}\left(M - \frac{p}{Q}\right), \quad \text{sive} \quad s\partial M - n\partial s\left(M - \frac{p}{Q}\right) = 0,$$

quae aequatio integrabilis evadit divisa per s^{n+1}, ut sit

$$\frac{s\partial M - nM\partial s}{s^{n+1}} = -\frac{np\partial s}{Qs^{n+1}},$$

1) Vide notam 1, p. 320. H. D.

cuius integrale est

$$\frac{M}{s^n} = -n\int\frac{p\,\partial s}{Q\,s^{n+1}}.$$

Quamobrem si ut ante ponamus

$$\int\frac{p\,\partial s}{Q\,s^{n+1}} = T\,,$$

habebimus $M = -nTs^n$, ideoque pro hac parte erit

$$v = -Tx^n s^n \quad \text{sive} \quad v = -Tz^n,$$

posito scilicet $z = xs$.

12. Pro altera parte litteram N involvente ea ob $xs = Z$ erit $N\frac{x^{n-1}}{s}\partial z$; quare cum N sit functio adhuc indeterminata, huius partis integrale erit functio quaecunque ipsius z, quae si designatur per Z, existente $\partial Z = Z'\,\partial z$ erit

$$N = \frac{sZ'}{x^{n-1}},$$

quocirca totum integrale erit

$$v = -Tz^n + Z.$$

Tum autem erit

$$\Theta = \frac{Z'z}{x^n} - nT\,s^n,$$

hincque colligitur ipsa functio quaesita

$$\Pi = \frac{\Theta Q - p}{qt - \Theta P},$$

quae solutio perfecte congruit cum praecedente.

13. Quanquam autem haec solutio totum negotium felicissime absolvit, tamen dantur casus, ad quos hanc solutionem vix ac ne vix quidem accommodare licet. Hoc scilicet evenit, quoties exponens $n = 0$, quoniam formulae $\int x^{n-1}\,\partial x$ valor tum est lx, quam ob causam iste casus peculiarem evolutionem postulat. Praeterea vero etiam casus, quo $Qqt - Pp = 0$, in superiore solutione non comprehenditur, quoniam posuimus

$$\frac{\partial s}{s} = \frac{Qq\,\partial t}{Qqt - Pp}.$$

Hanc igitur ob rem etiam hunc casum seorsim evolvi conveniet.

EVOLUTIO CASUS QUO $n = 0$

14. Hic igitur est

$$\partial v = \frac{p\partial x + \Pi q \partial y}{\Pi P + Q}\frac{1}{x},$$

quae aequatio, eliso ∂y, abit in hanc:

$$\partial v = \frac{\partial x}{x}\frac{p + \Pi q t}{\Pi P + Q} + \frac{\Pi q \partial t}{\Pi P + Q}.$$

Nunc ponatur ut ante

$$\Pi = \frac{\Theta Q - p}{qt - \Theta P},$$

et habebitur

$$\partial v = \frac{\Theta \partial x}{x} + \frac{q\partial t(\Theta Q - p)}{Qqt - Pp},$$

quae aequatio posito

$$\frac{Qq\partial t}{Qqt - P} = \frac{\partial s}{s} \quad \text{et} \quad \Theta = M + N$$

fit

$$\partial v = \frac{M\partial x}{x} + \frac{\partial s}{Qs}(MQ - p) + N\left(\frac{\partial x}{x} + \frac{\partial s}{s}\right).$$

Hic primum patet, prius membrum integrabile esse non posse, nisi sit M constans, tum autem comprehendi poterit in altero membro; quamobrem hic statim ponere licet $M = 0$, hincque ex priore parte fiet

$$v = -\int \frac{p\partial s}{Qs},$$

ita ut, posito

$$\int \frac{p\partial s}{Qs} = T,$$

hinc fiat $v = -T$. Pro altera autem parte, si statuamus ut ante $sx = z$, fit

$$\partial v = \frac{N}{z}\partial z.$$

Sit igitur

$$N = Z'z,$$

at $v = Z$, quocirca ob $M = 0$ erit

$$\Theta = Z'z,$$

hincque

$$\Pi = \frac{QZ'z - p}{qt - PZ'z},$$

atque hinc integrale completum erit $v = Z - T$, quae forma ex solutione generali deduci potuisset, at vero ex praesente casu promptius colligitur.

EVOLUTIO CASUS QUO $Qqt - Pp = 0$

15. Hoc igitur casu erit

$$P = \frac{Qqt}{p},$$

unde aequatio nostra fit

$$\partial v = \frac{x^{n-1}\partial x(p + \Pi qt) + x^n \Pi q \partial t}{\Pi Qqt + Qp} \cdot p,$$

quae contrahitur in hanc formam:

$$\partial v = x^{n-1}\partial x \frac{p}{Q} + \frac{x^n \Pi pq \partial t}{Q(\Pi qt + p)}.$$

Ponatur nunc

$$\frac{\Pi pq}{\Pi qt + p} = \Sigma,$$

ita ut sit

$$\Pi = \frac{\Sigma p}{q(p - \Sigma t)},$$

eritque

$$\partial v = \frac{p}{Q} x^{n-1}\partial x + \left(\frac{M\partial t}{Q} + \frac{N\partial t}{x^n Q}\right) x^n$$

ponendo scilicet

$$\Sigma = M + \frac{N}{x^n}.$$

16. Hic igitur prioris partis integrale erit $v = \frac{px^n}{nQ}$, si modo sit

$$M = \frac{Q}{n\partial t} \partial \cdot \frac{p}{Q}.$$

Pars vero posterior statim dat

$$v = \int \frac{N\partial t}{Q}.$$

Sit igitur

$$\int \frac{\partial t}{Q} = u,$$

ac si U denotet functionem quamcunque ipsius u, sumto $N = U'$ erit ex utraque parte

$$v = \frac{px^n}{nQ} + U$$

et nunc erit

$$\Sigma = \frac{Q}{n\partial t}\partial\cdot\frac{p}{Q} + \frac{U'}{x^n} \quad \text{et} \quad \Pi = \frac{\Sigma p}{q(p-\Sigma t)}.$$

17. Si fuerit $n = 0$, introducta littera Σ erit

$$\partial v = \frac{p\,\partial x}{Qx} + \frac{\Sigma\partial t}{Q},$$

unde si statuamus

$$\frac{\partial t}{Q} = \partial u,$$

ponamusque

$$\Sigma = Mlx + N,$$

denotante N functionem quamcunque ipsius u, scilicet $N = U'$, statim oritur ista aequatio integralis

$$v = \frac{p}{Q}lx + U,$$

siquidem fuerit

$$\partial\cdot\frac{p}{Q} = M\partial u.$$

Cum igitur sit

$$M = \frac{1}{\partial u}\partial\cdot\frac{p}{Q},$$

erit

$$\Sigma = \frac{lx}{\partial u}\partial\cdot\frac{p}{Q} + U',$$

unde ex praecedente formula functio quaesita Π colligitur

$$\Pi = \frac{\Sigma p}{q(p-\Sigma t)},$$

unde patet, istum casum prorsus diversae esse naturae, quam ut ex praecedente solutione deduci potuisset.

INTEGRATION D'UNE ESPECE REMARQUABLE D'EQUATION DIFFERENTIELLE DANS L'ANALYSE DES FONCTIONS A DEUX VARIABLES

Présenté à l'Académie le 11 Décembre 1777

Commentatio 785 indicis Enestroemiani

Mémoires de l'académie des sciences de St.-Pétersbourg 11, 1830, p. 131—137

Soit z une fonction des deux variables x et y et qu'on en tire les formules suivantes:

$$P = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}$$

$$Q = \frac{\partial\partial z}{\partial x^2} + \frac{2\,\partial\partial z}{\partial x\,\partial y} + \frac{\partial\partial z}{\partial y^2}$$

$$R = \frac{\partial^3 z}{\partial x^3} + \frac{3\,\partial^3 z}{\partial x^2\,\partial y} + \frac{3\,\partial^3 z}{\partial x\,\partial y^2} + \frac{\partial^3 z}{\partial y^3}$$

$$S = \frac{\partial^4 z}{\partial x^4} + \frac{4\,\partial^4 z}{\partial x^3\,\partial y} + \frac{6\,\partial^4 z}{\partial x^2\,\partial y^2} + \frac{3\,\partial^4 z}{\partial x\,\partial y^3} + \frac{\partial^4 z}{\partial y^4}$$

et ainsi de suite.

Cela posé je donnerai ici une méthode tout à fait singulière de trouver par une seule intégration l'intégrale complette de cette équation différentielle:

$$Az + BP + CQ + DR + ES + \text{etc.} = 0$$

à quelque degré que les différentielles puissent monter[1]).

1) Comparer p. 351 et p. 408 avec les mémoires 724 et 741 (d'après les numéros d'Eneström). Voir la note de la p. 407. H. D.

Pour cet effet il faut premièrement remarquer que toutes ces formules P, Q, R, S etc. tiennent un très beau rapport entr'elles; car comme on a

$$\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=P,$$

on trouvera

$$\frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}=\frac{\partial\partial z}{\partial x^2}+\frac{2\partial\partial z}{\partial x\partial y}+\frac{\partial\partial z}{\partial y^2}=Q$$

et de la même manière

$$\frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial y}=\frac{\partial^3 z}{\partial x^3}+\frac{3\,\partial^3 z}{\partial x^2\,\partial y}+\frac{3\,\partial^3 z}{\partial x\partial y^2}+\frac{\partial^3 z}{\partial y^3}=R.$$

Il y aura de même:

$$\frac{\partial R}{\partial x}+\frac{\partial R}{\partial y}=S,\quad \frac{\partial S}{\partial x}+\frac{\partial S}{\partial y}=T;$$

ces rapports nous donneront donc les égalités suivantes:

$$\text{I.}\quad \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=P,$$

$$\text{II.}\quad \frac{\partial P}{\partial x}+\frac{\partial P}{\partial y}=Q,$$

$$\text{III.}\quad \frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial y}=R,$$

$$\text{IV.}\quad \frac{\partial R}{\partial x}+\frac{\partial R}{\partial y}=S.$$

et ainsi de suite.

Après avoir remarqué ce beau rapport, je considère en général cette équation différentielle:

$$\frac{\partial v}{\partial x}+\frac{\partial v}{\partial y}=nv,$$

dont il s'agit de trouver l'intégrale complette. Pour cet effet je mets $\partial v=p\,\partial x+q\,\partial y$ et puisque

$$p=\frac{\partial v}{\partial x}\ \text{ et }\ q=\frac{\partial v}{\partial y}$$

cette équation différentielle prendra la forme suivante: $nv = p + q$ et partant $nv\partial y = p\,\partial y + q\,\partial y$, qui étant soustraite de l'équation supposée: $\partial v = p\,\partial x + q\,\partial y$ fournit celle-ci:

$$\partial v - nv\,\partial y = p(\partial x - \partial y)$$

qui, étant multipliée par e^{-nv} pour rendre le premier membre intégrable, donne

$$\partial \cdot ve^{-nv} = pe^{-nv}(\partial x - \partial y)$$

d'où l'on voit que le multiplicateur du dernier membre pe^{-nv} doit nécessairement être fonction de $x - y$ et alors son intégrale sera de même une telle fonction; par conséquent l'intégration nous fournit

$$ve^{-nv} = \mathfrak{A}:(x - y)$$

en employant la lettre $\mathfrak{A}$ pour marquer une fonction quelconque de la quantité qui y est jointe, et je me servirai dans la suite pour le même effet des lettres suivantes $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, pour en marquer d'autres fonctions. Voilà donc un beau lemme qui nous conduira à notre but proposé:

De cette équation différentielle: $nv = \frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}$ *l'intégrale complette est*

$$v = e^{+nv}\mathfrak{A}:(x - y).$$

Maintenant pour trouver l'intégrale en question supposons dans ce lemme $v = az + bP + cQ + dR$ prennant pour l'équation différentielle proposée celle-ci[1]):

$$Az + BP + CQ + DR + ES = 0$$

d'où l'on voit que la valeur de v doit renfermer un terme de moins que l'équation différentielle, et l'intégrale sera en vertu de notre lemme:

$$az + bP + cQ + dR = e^{nv}\mathfrak{A}:(x - y).$$

Qu'on met dans l'équation différentielle du lemme cette valeur prise pour v et on aura:

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1) Comparer avec la méthode indiquée p. 364. H. D.

$$naz + nbP + ncQ + ndR = + a\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) + b\left(\frac{\partial P}{\partial x} + \frac{\partial P}{\partial y}\right)$$
$$+ c\left(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial y}\right) + d\left(\frac{\partial R}{\partial x} + \frac{\partial R}{\partial y}\right).$$

Mettons donc ici au lieu des formules différentielles leurs valeurs finies marquées ci-dessus et notre équation tirée du lemme sera:

$$naz + nbP + ncQ + ndR = aP + bQ + cR + dS$$

qui étant rangée suivant l'ordre des lettres P, Q, R prendra cette forme:

$$naz + (nb - a)P + (nc - b)Q + (nd - c)R - dS = 0.$$

Donc puisque nous venons de trouver l'intégrale de cette équation:

$$az + bP + cQ + dR = e^{ny}\mathfrak{A}:(x - y),$$

on n'a qu'à rendre cette équation identique avec la proposée, savoir

$$Az + BP + CQ + DR + ES = 0$$

et nous aurons les égalités suivantes:

$$A = na,\ B = nb - a,\ C = nc - b,\ D = nd - c,\ E = - d,$$

d'où nous tirons les valeurs suivantes:

$$\begin{aligned}
d &= - E \\
c &= - nE - D \\
b &= - nnE - nD - C \\
a &= - n^3E - nnD - nC - B \quad \text{et enfin} \\
n^4E &+ n^3D + nnC + nB + A = 0.
\end{aligned}$$

Voilà donc une équation du quatrième ordre d'où l'on doit tirer la valeur de n, qui aura donc quatre valeurs que nous supposerons être α, β, γ, δ, dont chacune nous fournira une équation intégrale dont la première sera

$$az + bP + cQ + dR = e^{\alpha y}\mathfrak{A}:(x - y);$$

les autres valeurs β, γ, δ produisent aussi d'autres valeurs pour les lettres a, b, c, d, que nous distinguerons à la manière usitée et au lieu de $\mathfrak{A}$ nous employerons les autres charactères pour les fonctions de $(x - y)$; cela posé ces autres racines fourniront les équations intégrales suivantes:

$$\begin{aligned} a' z + b' P + c' Q + d' R &= e^{\beta v} \mathfrak{B}: (x - y) \\ a'' z + b'' P + c'' Q + d'' R &= e^{\gamma v} \mathfrak{C}: (x - y) \\ a''' z + b''' P + c''' Q + d''' R &= e^{\delta v} \mathfrak{D}: (x - y). \end{aligned}$$

De ces quatre équations il ne sera pas difficile de déduire les valeurs des quatre quantités z, P, Q, R.

Or il est évident que chacune de ces lettres sera exprimée par de certains multiples des quatre formules à la droite; mais nous n'en avons besoin que de la première z; donc puisque les multiplicateurs constans ne changent point les fonctions arbitraires nous n'en tiendront compte non plus et partant nous aurons pour z la valeur suivante

$$z = e^{\alpha v} \mathfrak{A}: (x - y) + e^{\beta v} \mathfrak{B}: (x - y) + e^{\gamma v} \mathfrak{C}: (x - y) + e^{\delta v} \mathfrak{D}: (x - y)$$

qui renfermant quatre fonctions arbitraires exprimera l'intégrale complette de l'équation différentielle proposée, que nous avons supposée monter au quatrième degré, quoiqu'il est facile à voir quelle sera l'intégrale pour les cas où l'équation proposée monteroit ou à un plus haut degré de différentielle ou à un plus bas.

Tout revient donc à resoudre cette équation algébrique:

$$A + nB + n^2 C + n^3 D + n^4 E + n^5 F + \text{etc.} = 0$$

dont les racines étant supposées α, β, γ, δ, etc. on sera d'abord en état d'assigner l'intégrale complette de toutes ces équations différentielles à quelque degré différentiel qu'elles puissent monter.

Cependant ils se pourront rencontrer des cas, où l'évolution de l'intégrale causeroit quelque difficulté, tels par exemple, où deux ou plusieurs des racines pour le nombre n seroient imaginaires ou égales entr'elles. Pour le premier cas supposons que les deux racines α et β soyent imaginaires et qu'on ait trouvé $\alpha = \mu + \nu\sqrt{-1}$ et $\beta = \mu - \nu\sqrt{-1}$ et pour déterminer réellement les deux membres de l'intégrale $e^{\alpha v} \mathfrak{A}: (x - y) + e^{\beta v} \mathfrak{B}: (x - y)$ posons

$$\begin{aligned} \mathfrak{A}: (x - y) &= \mathfrak{H}: (x - y) + \mathfrak{G}: (x - y) \quad \text{et} \\ \mathfrak{B}: (x - y) &= \mathfrak{H}: (x - y) - \mathfrak{G}: (x - y) \end{aligned}$$

et nous parviendrons à cette forme:

$$e^{\mu y}\,\mathfrak{H}:(x-y)\,(e^{\nu y\sqrt{-1}}+e^{-\nu y\sqrt{-1}})$$
$$+\,e^{\mu y}\,\mathfrak{G}:(x-y)\,(e^{\nu y\sqrt{-1}}-e^{-\nu y\sqrt{-1}}).$$

Or on sait par la réduction des imaginaires qu'il y a

$$e^{\nu y\sqrt{-1}}+e^{-\nu y\sqrt{-1}}=2\cos\nu y \quad \text{et} \quad e^{\nu y\sqrt{-1}}-e^{-\nu y\sqrt{-1}}=2\sqrt{-1}\,\sin\nu y$$

donc puisqu'on peut rejetter les facteurs constans les deux membres qui répondoient aux deux valeurs α et β se réduiront à cette forme réelle:

$$e^{\mu y}\cos\nu y\,\mathfrak{H}:(x-y)+e^{\mu y}\sin\nu y\,\mathfrak{G}:(x-y).$$

Pour l'autre cas où deux ou plusieurs des racines α, β, γ deviennent égales entre elles supposons d'abord $\beta=\alpha$ et puisqu'alors les deux premiers membres se réuniroient dans un seul et qu'on n'auroit plus autant de fonctions arbitraires que le degré de l'équation différentielle proposée exige, supposons $\beta=\alpha+\omega$ en prennant ω pour marquer une quantité infiniment petite, et puisque $e^{\omega y}=1+\omega y+\frac{1}{2}\omega^2 yy+$ etc., nous aurons $e^{\beta y}=e^{\alpha y}(1+\omega y)$, et puisqu'il est permis de mettre $\mathfrak{B}:(x-y)$ au lieu de $\omega\mathfrak{B}:(x-y)$, nous aurons au lieu des deux premiers termes qui répondent à α et β ces deux nouveaux $e^{\alpha y}\,\mathfrak{A}:(x-y)+e^{\alpha y}\,y\,\mathfrak{B}:(x-y)$. Par le même raisonnement on se convaincra facilement que, s'il y avoit trois racines égales $\alpha=\beta=\gamma$, on auroit au lieu des trois membres qui répondent à ces lettres ces trois autres:

$$e^{\alpha y}\,\mathfrak{A}:(x-y)+e^{\alpha y}\,y\,\mathfrak{B}:(x-y)+e^{\alpha y}\,y^2\,\mathfrak{C}:(x-y)$$

et s'il y avoit une quatrième racine égale, on n'auroit qu'à ajouter aux trois termes annoncés ce quatrième $e^{\alpha y}\,y^3\,\mathfrak{D}:(x-y)$, d'où nous pourrons resoudre les problèmes particuliers suivans.

PROBLEME I

Trouver l'intégrale complette de cette équation particulière:

$$P=0,\quad ou\quad \frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=0.$$

SOLUTION

Puisque ici $P = 0$, nous aurons dans l'équation générale:

$$A = 0, \quad B = 1, \quad C = D = E = 0,$$

d'où l'équation pour trouver le nombre n sera $n = 0$; d'où l'on tire $\alpha = 0$ et partant l'intégrale complette sera $z = \mathfrak{A}: (x - y)$.

PROBLEME II

Trouver l'intégrale complette de cette équation $Q = 0$ ou bien

$$\frac{\partial\partial z}{\partial x^2} + \frac{2\,\partial\partial z}{\partial x \partial y} + \frac{\partial\partial z}{\partial y^2} = 0\,.$$

SOLUTION

Puisque ici $Q = 0$, nous aurons dans la formule générale:

$$A = 0, \quad B = 0, \quad C = 1, \quad D = E = F = 0\,;$$

d'où pour déterminer le nombre n, nous aurons cette équation $nn = 0$, donc les deux racines $\alpha = 0$ et $\beta = 0$ et partant égales entr'elles; par conséquent l'intégrale complette sera:

$$z = \mathfrak{A}: (x - y) + y\,\mathfrak{B}: (x - y).$$

PROBLEME III

Trouver l'intégrale complette de cette équation:

$$\frac{\partial^3 z}{\partial x^3} + \frac{3\,\partial^3 z}{\partial x^2 \partial y} + \frac{3\,\partial^3 z}{\partial x \partial y^2} + \frac{\partial^3 z}{\partial y^3} = 0\,.$$

SOLUTION

Puisque ici $R = 0$, nous aurons dans l'équation générale:

$$A = 0, \quad B = 0, \quad C = 0, \quad D = 1\,, \quad E = F = 0\,;$$

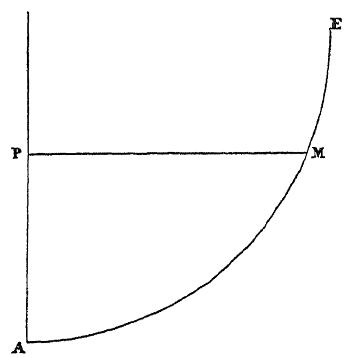
d'où l'équation pour le nombre n sera $n^3 = 0$ et partant $\alpha = \beta = \gamma = 0$; par conséquent l'intégrale complette sera:

$$z = \mathfrak{A}: (x - y) + y\,\mathfrak{B}: (x - y) + y^2\,\mathfrak{C}: (x - y).$$

FRAGMENTUM EX ADVERSARIIS MATHEMATICIS DEPROMTUM

Commentatio 856 indicis ENESTROEMIANI

Opera postuma 2, Petropoli 1862, p. 824—826



PROBLEMA

Si corpus super curva ascendat in medio resistente, fueritque $AP = x$ et $AM = s$, celeritas vero in $M = u$, et resistentiae formula $= \varDelta + bu + cuu$, ita ut sit $udu + dx + (\varDelta + bu + cuu)ds = 0$, invenire aequationem inter x et s, ut ista aequatio resolutionem admittat.

SOLUTIO

Primo quidem patet hanc aequationem resolubilem fieri casu $dx = ads$; tum enim statim habetur

$$ds = \frac{-udu}{a + \Delta + bu + cuu}.$$

Praeter hunc vero casum difficile est alios invenire; unde sequens solutio eo magis est notatu digna. Ponatur $ds = \frac{dq}{f - cq}$, ac statuatur

$$dx = \frac{aqdq - \Delta dq}{f - cq};$$

tum enim aequatio induet hanc formam:

$$(f - cq)udu + aqdq + budq + cuudq = 0.$$

Hic ponatur $u = qv$, ac prodibit

$$\frac{dq}{q(f - cq)} = \frac{-vdv}{a + bv + fvv},$$

sicque quantitas q per v definiri poterit; tum vero etiam $u = qv$ per v definitur, at vero x et s dantur per q. Hinc iam porro tempus assignari poterit ex formula

$$dt = \frac{ds}{u} = \frac{dq}{(f - cq)qv},$$

unde si loco

$$\frac{dq}{q(f - cq)}$$

eius valor substituatur, erit

$$dt = \frac{-dv}{a + bv + fvv}.$$

Sic tempus t etiam functioni ipsius v aequabitur $= \Gamma : \frac{u}{q}$, scilicet t aequabitur functioni nullius dimensionis binarum q et u. Unde patet totum tempus ascensus fore constans. Incipit enim ubi $s = 0$, at s ita definiri potest per q, ut posito $s = 0$, fiat $q = 0$; ergo in determinatione temporis integrale ita sumi debet, ut evanescat sumto $q = 0$. Ascensus vero terminatur, ubi celeritas u evanescit, unde totum tempus ascensus reperitur ex integrali invento ponendo $u = 0$, quod propterea erit quantitas constans, quaecunque fuerit celeritas initialis in A. Evidens igitur est curvam hoc modo inventam simul esse *tautochronam* in hoc medio resistente.

PROBLEMA

Proposita aequatione differentiali

$$udu + dx + ds(a + bu + cuu) = 0,$$

quaeritur qualis functio ipsius s loco x assumi debeat, ut ista aequatio resolutionem admittat.

SOLUTIO

Ponatur $dx + ads = Sds$, ut habeatur haec aequatio:

$$udu + ds(S + bu + cuu) = 0.$$

Haec aequatio fingatur integrabilis reddi, si dividatur per hanc formulam $Aqq + Bqu + Cuu$, ubi q designet certam functionem ipsius s. Quare cum in genere formula $Pdu + Qds$ integrationem admittat, si fuerit

$$\left(\frac{dP}{ds}\right) = \left(\frac{dQ}{du}\right),$$

pro nostro casu erit:

$$P = \frac{u}{Aqq + Bqu + Cuu} \quad \text{et} \quad Q = \frac{S + bu + cuu}{Aqq + Bqu + Cuu}.$$

Iam quia q supponitur functio ipsius s, ponatur $dq = rds$, ac reperietur:

$$\left(\frac{dP}{ds}\right) = \frac{-2Aqru - Bruu}{(Aqq + Bqu + Cuu)^2},$$

deinde

$$\left(\frac{dQ}{du}\right) = \frac{Abqq + 2Acqqu - BSq + Bcquu - 2CSu - Cbuu}{(Aqq + Bqu + Cuu)^2}.$$

Hinc igitur orietur ista aequatio:

$$Abqq - BSq + 2Aqu(r + cq) - 2CSu + Bcquu + Bruu - Cbuu = 0,$$

quorum trium membrorum singula ad nihilum redigi debent. Ex primo fit:

$$Abqq - BSq = 0, \quad \text{unde} \quad S = \frac{Abq}{B}.$$

Secundum membrum dat $2Aqur + 2Acqqu - 2CSu = 0$, quae loco S substituto valore abit in:

$$2Aqr + 2Acqq - \frac{2ACbq}{B} = 0, \quad \text{unde} \quad r = \frac{Cb - Bcq}{B}.$$

Denique tertium membrum praebet $Bcq + Br - Cb = 0$, unde iterum prodit $r = \frac{Cb - Bcq}{B}$, qui ambo valores ipsius r cum sint inter se aequales, nihil amplius determinandum restat, quare cum $r = \frac{dq}{ds}$, habebitur

$$\frac{dq}{ds} = \frac{Cb - Bcq}{B} \quad \text{hincque} \quad ds = \frac{Bdq}{Cb - Bcq}.$$

Hincque iam fiet

$$cs = \int \frac{Bcdq}{Cb - Bcq} = l \frac{\varDelta}{Cb - Bcq}.$$

Quodsi iam velimus, ut sumto $s = 0$ etiam q evanescat, esse debet $\varDelta = Cb$, sicque habebitur:

$$cs = l \frac{Cb}{Cb - Bcq}, \text{ ideoque } e^{cs} = \frac{Cb}{Cb - Bcq}, \text{ unde fit } q = \frac{Cb(e^{cs} - 1)}{Bce^{cs}} = \frac{Cb}{Bc}(1 - e^{-cs}).$$

Valore autem ipsius q invento, erit $S = \frac{ACbb}{cBB}(1 - e^{-cs})$; quare cum sit:

$$-dx + ads = Sds, \quad \text{erit} \quad x = \int Sds - as = \frac{ACbb}{cBB}\left(s + \frac{1}{c}e^{-cs}\right) - as + \text{Const.}$$

Unde ut sumto $s = 0$ fiat $x = 0$, fiet:

$$-\text{Const.} = -\frac{ACbb}{ccBB}, \quad \text{consequenter} \quad x = -\frac{ACbb}{ccBB} - as + \frac{ACbb}{cBB}\left(s + \frac{1}{c}e^{-cs}\right).$$

Hic ergo litterae A, B, C penitus arbitrio nostro relinquuntur; interim tamen patet, statui non posse $B = 0$; tum vero fieri debet $Cb > Bcq$, donec q certum valorem obtinuerit, in quo terminus scilicet q maximum habet valorem. Invento autem divisore $Aqq + Bqu + Cuu$ integratio aequationis nulla amplius laborat difficultate, scilicet per logarithmos et arcus circulares.

PROBLEMA

Proposita aequatione

$$udu + dx + (a + bu + cuu)ds = 0,$$

ubi x sit certa functio ipsius s, invenire valorem formulae integralis $t = \int \frac{ds}{u}$.

SOLUTIO

Ponatur brevitatis gratia

$$udu + dx + (a + bu + cuu)ds = dW,$$

ut sit $dW = 0$. Iam semper certus multiplicator M dabitur, ut formula $\frac{ds}{u} + MdW$ integrationem admittat. Cum igitur sit $dW = 0$, erit $dt = \frac{ds}{u} + MdW$; quae formula cum sit integrabilis, inde definietur ipsum tempus t. Sumatur

$$M = \frac{p}{u(\alpha qq + \beta qu + \gamma uu)},$$

ubi p et q sint certae functiones ipsius s, eritque:

$$dt = \frac{ds}{u} + \frac{pudu + pdx + pads + pbuds + cpuuds}{u(\alpha qq + \beta qu + \gamma uu)},$$

hinc ergo numerator erit

$$\alpha qqds + \beta quds + \gamma uuds + pudu + pdx + apds + bpuds + cpuuds,$$

denominatore existente $u(\alpha qq + \beta qu + \gamma uu)$. Nunc autem fiat

$$dt = \frac{qdu - udq}{\alpha qq + \beta qu + \gamma uu},$$

quippe quae formula semper potest integrari; ponatur enim $u = qv$, eritque

$$dt = \frac{dv}{\alpha + \beta v + \gamma vv}.$$

Facta autem hac aequalitate:

$$\alpha qqds + \beta quds + \gamma uuds + pudu + pdx + apds + bpuds + cpuuds = qudu - uudq,$$

fieri debet:

$$\alpha qqds + pdx + apds = 0,\ p = q,\ \beta quds + bpuds = 0,\ \gamma uuds + cpuuds = -uudq^{1)};$$

unde

$$ds = \frac{-dq}{\gamma + cp} \quad \text{et} \quad dx = -\frac{(ap + \alpha qq)ds}{p} = -(a + \alpha p)ds.$$

Ubi notetur, cum posito $u = qv$ sit

$$dt = \frac{dv}{\alpha + \beta v + \gamma vv},$$

ideoque

$$t = f : v = f : \frac{u}{q},$$

ideoque t aequabitur functioni nullius dimensionis ipsarum q et u; quare totum tempus exprimetur numero absoluto, ideoque curva erit *tautochrona.*

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1) Hinc oriuntur duae aequationes praebentes p. Ut eundem pro p valorem afferent, habebitur: $d^2x + cdxds + (ac - a\gamma)ds^2 = 0$, quae integrata dat: $x = \frac{a\gamma - ac}{c} s + k(1 - e^{-cs})$, sicut in problemate supra scripto. H. D.